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Volume 132

# Calculus of Variations and Harmonic Maps

Hajime Urakawa

American Mathematical Society

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# Calculus of Variations and Harmonic Maps

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Volume 132

# Calculus of Variations and Harmonic Maps

Hajime Urakawa

Translated by
Hajime Urakawa

American Mathematical Society
Providence, Rhode Island

# 変分法と調和写像

**HENBUNPŌ TO CHŌWA SHAZŌ**
**(Calculus of Variations and Harmonic Maps)**

by Hajime Urakawa



Translated from the Japanese by Hajime Urakawa

1991 *Mathematics Subject Classification.* Primary 58Exx;
Secondary 53Cxx, 58Bxx, 58Dxx, 58Gxx, 81Txx.

ABSTRACT. This book gives an introduction to and a comprehensive exposition of studies and recent developments in the general theory of calculus of variations and the theory of harmonic maps. It begins with an introduction that provides a perspective on variational methods and their applications to physics. A general theory of calculus of variations is given in Chapters 2 and 3 including the Morse theory on infinite-dimensional manifolds due to Palais-Smale and the existence of minimizing maps in each homotopy class. The last three chapters present an introduction to harmonic maps, Xin's instability theorem on harmonic maps from spheres and the stability theorem on holomorphic maps, and a survey of recent results on harmonic maps.

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# Contents

# Preface to the English Edition

The theory of variational methods, in particular, gauge theory and the theory of harmonic maps has developed explosively over the last decade. The theory is of essential importance in both physics and mathematics. In this theory, the notion of a manifold, particularly an infinite-dimensional manifold plays an essential role. However, a physicist colleague at my university said to me once that mathematics, especially the notion of "manifold", is difficult to learn. Every physicist wants to know mathematics, but every book on differential geometry begins with an explanation of the notion of "manifold". This is a hard obstacle for beginners. At the time, I could only reply that the earth is round and that to analyze it as lying in a flat plane is easy but would lead only to a theory appropriate to the time before Columbus. When the opportunity arose for me to write a book about harmonic maps, I recalled the above dialogue and decided to write a book in which the first chapter contains no definitions of mathematical notions, but rather contains an introduction that explains the importance of learning the notion of "manifold".

Such being the case, I started this book by giving a perspective of the roles of variational methods and infinite-dimensional manifolds in mathematics and in physics followed by elementary examples of physical problems including problems in classical mechanics.

Readers of this book, who are familiar with Riemannian geometry and want to get quickly to the theory of harmonic maps, can start with Chapter 4.

Logically, this book is constructed as indicated by the diagram on the next page.

In Chapter 2, the notions of a Banach manifold, Hilbert manifold, and the usual finite-dimensional manifold are introduced. Several notions in Riemannian geometry and several examples of Riemannian manifolds are given, and this chapter closes with the example of the infinite-dimensional manifold consisting of all $L_{1,p}$-maps from a compact manifold $M$ into another compact manifold $N$. This manifold, denoted by $L_{1,p}(M, N)$, is important in Chapters 3, 4, and 6.

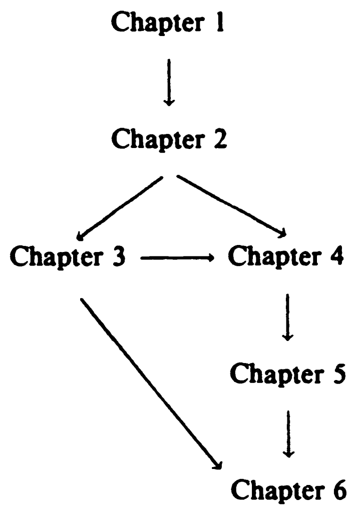
In Chapter 3, the Morse theory on Hilbert or Banach manifolds, which was initiated by Palais and Smale under the assumption of the Palais-Smale condition (C), is explained. It is proved that if $p > \dim M$, then the manifold $L_{1,p}(M, N)$ satisfies the condition (C).

In Chapter 4, the notion of harmonic maps and the first variation formula are introduced. Several examples of harmonic maps are presented.

The main topic in Chapter 5 is the second variation formula, and the notion of stability of harmonic maps is defined. Xin's instability theorem is proved, and related results about the stability of holomorphic maps between Kähler manifolds are given along with their proofs.

This book closes with Chapter 6 whose main topics are the existence and construction problems of harmonic maps to the unit sphere, the complex projective space, the unitary group, or to a compact nonpositively curved Riemannian manifold.

I hope this book will be helpful to students of mathematics and to mathematical scientists who want to know and to study the recent developments in the theories of harmonic maps and the variational methods which have applications to broad areas of science.




Hajime Urakawa
Sendai
February 1993

# Preface

In ancient times, Queen Dido of Carthage ordered her subjects to enclose a maximum area of land making use of a given string made from the skin of a cow. This problem has been known as the isoperimetric problem and one of the origins of the calculus of variations. What did the Queen's subjects answer to her ?

As time went by, it was observed by J. Kepler, I. Newton, and G.W. Leibnitz that the laws of nature can be described in terms of differential equations. In particular, the law of universal gravitation due to Newton became the origin of the differential calculus.

In the middle of the eighteenth century, L. Euler and J.L. Lagrange found that the equation due to Newton is induced from "the problem of maximum and minimum". This was the origin of variational calculus and analysis. Since then, Cauchy, Weierstrass, and Fourier established the foundations of the differential calculus. At present, in high schools and the first year of college, one studies differential and integral calculus from these historical perspectives.

In the nineteenth century, Lobachevsky, Bolyai, and Gauss discovered the notion of non-Euclidean spaces, and Riemann discovered the notion of Riemannian metrics, and the differential and integral calculus of curved manifolds, that is, differential geometry. In the early part of the twentieth century, A. Einstein's theory of general relativity led to an increase in interest in Riemannian geometry and in differential geometry.

The word "Geometry" originally meant measuring "Geo = the earth", the origin of geometry is geodesy. A geodesic is, roughly speaking, a shortest curve between two points. This means that one should treat the notion of "shortest curve" in the space of all curves. This is a typical model of the calculus of variations. The space of all curves is a very big space of infinite dimension, and the differential calculus and the calculus of variations due to Newton, Euler, and Lagrange allowed the study of calculus in an infinite-dimensional space. Several principles in physics can be described by this fundamental idea in the calculus of variations, which is the search for the minima of some function such as the length.

M. Morse began the study of the relation between critical points of a function on a curved manifold and the topology of the manifold by making use of these studies about geodesics. But to systematically study several variational problems one had to treat infinite-dimensional spaces which were also important in the quantum field theory due to Dirac and Heisenberg. Furthermore , the notion of a curved infinite–dimensional space, a so-called "infinite-dimensional manifold" was necessary to handle a space of curves in a satisfactory manner. From the latter half of the 1950's to the early 1960's, J. Eells, R.S. Palais , and S. Smale showed the space of all smooth mappings to be an infinite-dimensional manifold, and established a general theory of differentiation of smooth functions on it and a critical point theory of functions. The notion of harmonic mappings was established as critical points of the energy (the action integral), and the existence of a harmonic mapping into a nonpositive curved manifold was shown by Eells and Sampson.

Another origin of the theory of harmonic mappings was a problem proposed by J. Plateau in the nineteenth century. This is a problem concerning the existence and uniqueness of soup bubbles bounding a given wire (a closed Jordan curve) in the 3-dimensional Euclidean space, in other words, to search for a surface minimizing the area among all surfaces bounding a given Jordan curve. T. Rado and J. Douglas solved this problem independently in 1930 ~ 1931 . C.B. Morrey solved Plateau's problem for an arbitrary Riemannian manifold in 1948. These results can be regarded as theories of harmonic mappings defined on a two-dimensional region with the boundary conditions.

Today the theory of harmonic mappings is one of the most important theories in areas of geometry such as the theories of Einstein metrics (= theory of gravitations), Yang-Mills connections (= gauge theories), and it has many applications in several fields.

It has been known for some time that there are difficulties in applying the theory of Palais and Smale to several variational problems, for instance, the theory of harmonic mappings. In particular, to apply the theory we need the condition (C) of Palais and Smale. For condition (C) to be satisfied we need the boarderline estimate of Sobolev's Lemma which is one of the fundamental tools in analysis. So one can not apply variational methods which do not satisfy condition (C). Many interesting geometric problems, the theories harmonic mappings and Yang-Mills connections do not satisfy condition (C). So one of the crucial objectives is to find a detour that bypasses condition (C). K. Uhlenbeck succeeded in 1981 in finding a method that gives an alternative proof of the Eells and Sampson's theorem about the existence of harmonic mappings into nonpositively curved manifolds, and she proved, independently of L. Lemaire, the existence of a harmonic map of a 2-dimensional manifold without boundary into a target manifold $N$ with $\pi_2(N) = \{0\}$ , which is the analogue of Plateau's Problem. But in the case of a domain manifold of higher dimension, the existence problem for harmonic maps is still unsolved, and there are a lot of unsolved interesting problems in the calculus of variations.

In this book, except for the "Coffee Breaks" in Chapters 1, 3, and 4, we have devoted the exposition to presenting a general theory of the calculus of variations and the theory of harmonic mappings. If one reads through this book, one obtains a fundamental knowledge of differential geometry and can start to do research on harmonic mappings after referring to the three vast works due to J. Eells and L. Lemaire. We hope this book becomes a good handbook to researchers working on harmonic mappings and the calculus of variations. This book is based on the author's lectures at Hokkaido University, Osaka University, Hiroshima University, Tokyo Institute of Technology, Ryukyu Univiversity, Tsukuba University, and on lectures in undergraduate courses at Tohoku University. During the time this book was being prepared at M. S. R. I., Berkeley, I was stimulated by a workshop related to the global analysis, algebraic topology, and quantum field theory which has been studied by Atiyah, Witten, Manin, Segal, Tsuchiya, Kanie, and others. We hope that reading through this book will also become a first step toward the study of these vast research areas.

Finally, I express my sincere gratitude to Professor Shingo Murakami and Mr. Shuji Hosoki of the Shokabo Publishing Co., Ltd. who encouraged me to publish this book and who have given many valuable suggestions during the preparation of this book.

Hajime Urakawa<br>Sendai<br>Fall 1990

CHAPTER 1

# Calculus of Variations

The calculus of variations is a theory based on the belief that it is possible to explain all things in the universe. In this chapter, we shall observe several important theories of mathematics and physics that are derived by variational methods and we will present some examples of the calculus of variations.

The aims of this book are to present a general theory of the calculus of variations, its applications, along with some serious difficulties with the calculus of variations and how to overcome these difficulties in applying the theory of *harmonic mappings*, which have been called *nonlinear sigma models* by theoretical physists. In this chapter, we give an outline.

## §1. The aims of this book

The aims of this book are, using a principle, called the variational method or the calculus of variations which is useful in natural sciences, to understand harmonic mappings (nonlinear sigma model) which are the most natural objects among smooth mappings and to introduce development of their theories.

| What is the method of variations, or the variational principle ? |
|---|

This is a method to select the best among a variety of objects. It is the following process (see Figure 1.1, next page):

(1) Gather all relevant objects into a space $X$.

(2) Take an appropriate function $E$ on $X$. If $E$ is appropriate for the purpose, then minima or maxima of $E$ in $X$ are the best objects.

Putting this idea into practice has been very difficult. However, a lot of people have been fascinated by this idea.

From the time of I. Newton, G.W. Leibnitz, P.L. Maupertuis, L. Euler, and J. L. Lagrange, the calculus of variations has been carried out as follows:

(1) On the space $X$, one may consider the concept of the *differential* $E'$ of $E$.

(2) Then if $x_0 \in X$ is *best*, then it should attain the maximum or minimum of $E$. So the *derivative* of $E$ vanishes at $x_0$, i.e.,

$$E'(x_0) = 0.$$

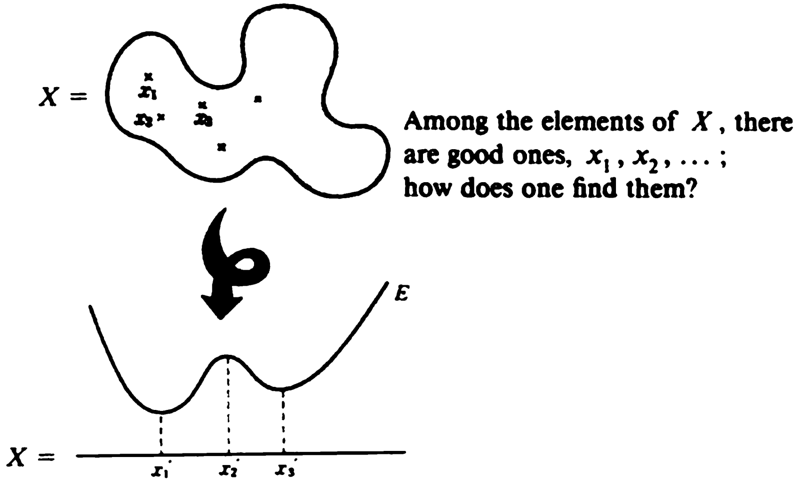



FIGURE 1.1

(3) The point $x_0$ satisfying $E'(x_0) = 0$ (called a **critical point** of $E$) could be written and characterized in term of some differential equation (called the **Euler-Lagrange equation**).

(4) Thus, it remains only to solve this differential equation.

Sometimes, we can trace the inverse of the above approach:

(1) One should solve some differential equation which is important but difficult to solve.

(2) To solve this equation, one could consider a certain space $X$ and a function $E$ on $X$ in such a way that the Euler-Lagrange equation corresponding to $E$, i.e., $E'(x_0) = 0$ corresponds to the equation in question.

(3) Then one may only find a maximum or minimum of $E$ on $X$.

For many interesting problems in mathematics and physics, one could formulate the calculus of variations in this way, but it often happens that both to find maxima and minima of $E$ and to solve the corresponding Euler-Lagrange equation are very difficult.

In the mid-1960s, R. Palais and S. Smale independently clarified under which conditions on $E$ and $X$, $E$ has minima. This condition is called the **Palais-Smale condition** (C).

To explain the Palais-Smale condition (C), we consider the following examples:

$$f(x) = x^2, \qquad -\infty < x < \infty, \tag{1}$$

$$g(x) = e^{x^3}, \qquad -\infty < x < \infty. \tag{2}$$

Both functions have infima 0, have only one critical point at $x = 0$, and tend

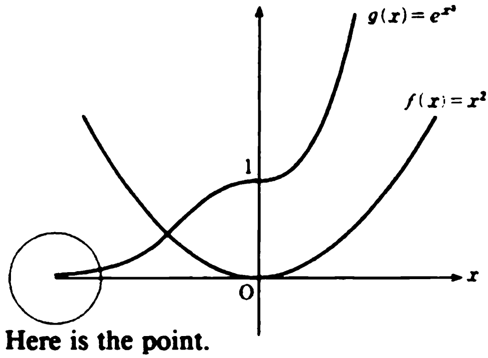



FIGURE 1.2

to infinity as $x \to \infty$. (See Figure 1.2.) But they are very different from each other. (1) has a minimum 0 at $x = 0$, but (2) does not attain any minimum. What is the reason for these phenomena ?

**One answer.** For (1), $f''(0) = 2 > 0$, but for (2), $g''(0) = 0$, $g'''(0) = 6$. (so-called a study the sign of the second derivative of a function.)

**Another answer.** The former satisfies Palais-Smale's condition (C), but the latter does not (see §2, Chapter 3).

The first answer can be formulated in a theory of stability of critical points, i.e., to study the sign of the second derivative $E''$ (the **second variation**, or the **Hessian**) of a function $E$ on $X$. The second answer, i.e., to check whether the Palais-Smale condition (C) holds or not is a litmus test as to whether a given variational problem is difficult or not. We shall show that if the condition (C) holds and the corresponding function $E$ is bounded below, then $E$ attains a mimimum, which gives the desired answer! Otherwise, the problems are very difficult. It so happens that many interesting problems arising from geometric problems do not satisfy condition (C), but $E$ has a minimum.

This book consists of the following chapters:

Chapter 1. We shall give several examples of the method of variations, calculate the first variation $E'$, and learn how to derive the Euler-Lagrange equations.

Chapter 2. Considering the derivative $E'$ of $E$ on $X$ is, strictly speaking, introducing a "manifold structure" on $X$ and a differentiable function $E$ on it, and considering the derivative $E'$. This chapter is the foundation for a rigorous treatment of the methods of variation.

Chapter 3. We shall explain the notions of critical point of a differentiable function $E$ on $X$, and the second derivative (the Hessian). We shall also explain the Palais-Smale condition (C) and show that if it is satisfied, then the function $E$ which is bounded below, attains a munimum. We shall present a theorem that holds when the condition (C) is satisfied.

Chapter 4. As applications of the above theory, we shall introduce the theory of harmonic mappings, called nonlinear sigma models by theoretical physicists. Chapters 4, 5, and 6 can be read independently of the previous chapters.

Chapter 5. We shall present the second variation formula for harmonic mappings, and the stability or unstability of harmonic mappings as applications.

Chapter 6. The theory of harmonic mappings does not, in general, satisfy the Palais-Smale condition (C). We shall explain Uhlenbeck's method to overcome this difficulty, and present recent developments on the existence, construction, and classification theories of harmonic mappings.

## §2. Methods of variations and field theories

Methods of variations are essentially important in physics, in particular, in the field theories. In this section, we shall give an overview of harmonic mappings and other related field theories.

It is known that there exist in nature, four kinds of forces — the gravitation, electromagnetism, weak interaction, and strong interaction. There have been attempts to join these forces in a unified field theory. Gravitation has been described as Einstein's general relativity theory, and electromagnetism as Maxwell's theory. These four forces are understood as gauge field theories by physicists. Recently, superstring theory, conformal field theory, and topological quantum field theory have received much attention in both mathematics and physics.

Mathematics, especially geometry, has been developed, sometimes deeply influenced by and sometimes independently of such developments in physics. In fact, some of the theories in physics correspond to theories in mathematics. See Table 1.1

TABLE 1.1

| Mathematics | Physics |
|---|---|
| Einstein metrics | Gravitations |
| Geometry of connections | Gauge theory |
| Harmonic maps | String theories (Nonlinear sigma models) |

Here, postponing any rigorous definitions, we describe these three theories as derived by methods of variations.

**Einstein metrics.** We take as the space $X$ the totality of Riemannian metrics with volume 1 on a fixed $m$-dimensional manifold, and as the function $E$ on $X$ we take

$$E(g) := \int_M S_g v_g \quad \text{(called the **total curvature** of } (M, g)), \qquad g \in X.$$

$S_g$ is the scalar curvature, and $v_g$ is the canonical measure given by

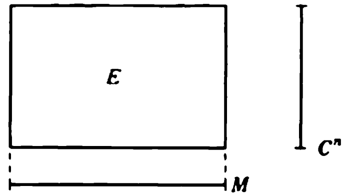



FIGURE 1.3

$v_g = \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_m$ for a Riemannian metric $g = \sum g_{ij}\, dx_i \otimes dx_j$. It is known (cf. [N], called **Hilbert's theorem**) that for any deformation $g_t$, $-\epsilon < t < \epsilon$, $g_0 = g$,

$$g \text{ is a critical point of } E \text{ in } X \iff \left.\frac{d}{dt}\right|_{t=0} E(g_t) = 0,$$
$$\iff g \text{ is an Einstein metric, i.e., } \rho = c\, g.$$

Here $\rho$ is the Ricci tensor of $g$, and $c$ is some constant.

**Yang-Mills connections.** Let $E$ be a vector bundle over a compact Riemannian manifold $(M, g)$, for example, $E = M \times \mathbf{C}^n$ (a direct product). See Figure 1.3.

Then for the a space $X$ we take , the totality of connections $\tilde{\nabla}$ of $E$, and as a function $E$ on $X$ we take

$$E(\tilde{\nabla}) := \frac{1}{2} \int_M \|R^{\tilde{\nabla}}\|^2 v_g, \quad \tilde{\nabla} \in X.$$

Here $R^{\tilde{\nabla}}$ is the curvature tensor of $\tilde{\nabla}$ of $E$, and $\|\cdot\|$ is its norm (see 1.3 in Chapter 4). Then by definition for any deformation $\tilde{\nabla}_t$ of $\tilde{\nabla}$, $-\epsilon < t < \epsilon$, $\tilde{\nabla}_0 = \tilde{\nabla}$,

$$\tilde{\nabla} \text{ is a critical point of } E \iff \left.\frac{d}{dt}\right|_{t=0} E(\tilde{\nabla}_t) = 0,$$
$$\iff \tilde{\nabla} \text{ is a Yang-Mills connection.}$$

**Harmonic mappings.** We consider two compact Riemannian manifolds $(M, g)$, $(N, h)$ and take for $X$ the set $C^\infty(M, N)$ of all smooth mappings of $M$ into $N$. For the function $E$ on $X$, we take

$$E(\phi) := \frac{1}{2} \int_M \|d\phi\|^2 v_g, \qquad \phi \in X = C^\infty(M, N),$$

where $\|d\phi\|$ is the norm of the differential $d\phi$ of a mapping $\phi \in C^\infty(M, N)$ with respect to the metrics $g$, $h$. (See Figure 1.4, next page.) Then by definition (see 1.2, Chapter 4) that for any deformation $\phi_t$ of $\phi$, $-\epsilon < t < \epsilon$, $\phi_0 = \phi$,

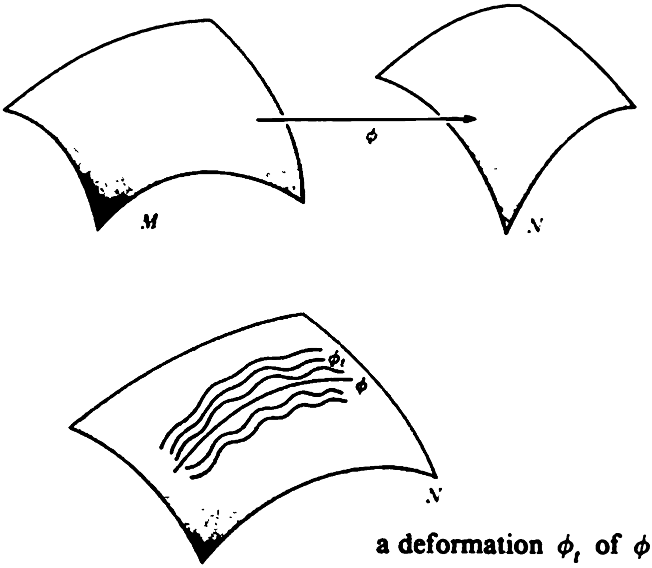



FIGURE 1.4

$$\phi \text{ is a critical point of } E \iff \left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0,$$
$$\iff \phi \text{ is a harmonic map},$$
i.e., a nonlinear sigma model.

It is known that there are several mysterious and strong similarities among these three theories. See Table 1.2.

TABLE 1.2

| | Einstein metric | Yang-Mills conn. | Harmonic map |
|---|---|---|---|
| Stable | nothing | (A.) S.D. conn. | holom. maps |
| Moduli | E-met. deform. | moduli of S.D. | Calabi constr. |
| Homog. | invar. E-met. | invar. Y.M. | minimal orbits |
| O.D.E. | inhomo. E-met. | non-S.D. Y.M. | O.D.E. constr. |

In Table 1.2, "Stable" means stability or minimality of $E$; "nothing" indicates that every Einstein metric is unstable, but Einstein-Kähler metrics have some stabilities; "(A.) S. D." stands for (anti)selfdual connections, and "holom. maps" stands for holomorphic maps between Kähler manifolds (cf. §3, Chapter 5)

"Moduli" stands for theories of moduli spaces or deformation theories; "E-met. deform.", stands for deformation theories of Einstein metrics; "moduli of S.D." stands for moduli theories of (anti)selfdual connections, and "Calabi constr." stands for the Calabi construction of harmonic mappings from 2-spheres into symmetric spaces (cf. 2.3 Chapter 6).

"Homog." stands for homogeneity; "invar. E-met." stands for group action invariant Einstein metrics on homogeneous spaces; "invar. Y.M." stands for group action invariant Yang-Mills connections on homogeneous spaces, and "minimal orbits" stands for the theory of homogeneous minimal submanifolds (cf. [H.L], [T.T], [M.O.U]).

"O.D.E." stands for the theories of ordinary differential equations arising from equivariant theories; "inhomo. E-met." stands for the constructions of inhomogeneous Einstein metrics; "non-S.D. Y.M.", stands for the constructions of unstable non-(anti)selfdual Yang-Mills connections; "O.D.E. constr." stands for the theory of constructions of harmonic mappings using by ordinary differential equations(cf. 2.4, Chapter 6). (This table could be continued.)

A moral obtained from Table 1.2 is that if one finds an interesting result in any box of these three theories, then one can expect to find analogues in the others! This suggests a possibility of the existence of a unified field theory in physics.

This book is an exposition introducing the methods of variations and focussing on the theory of the last vertical line in Table 1.2, harmonic mappings.

## §3. Examples of the method of variations

In this section, we shall introduce examples of the "method of variations" known classically and show how to derive the Euler-Lagrange equations. The calculations in this section are in the textbooks for the first course in physics at many universities. We recommend that the reader refer also these textbooks.

**3.1. Equation of Equilibrium States of Strings.** Let us consider a homogeneous elastic string put first on the closed interval $[0, L]$ in the $xy$-plane and not pressed by any external forces. See Figure 1.5, next page. Next we assume an external force acting in the direction $y$-axis and with strength $f(x)$ at each point $x \in [0, L]$. In this case, we consider a problem finding the position of **equilibrium state of this string**. Here our assumptions for a string being homogeneous and elastic mean the force of tension is constant $\mu$, and the density is constant $\rho$.

We denote by $u(x)$, $x \in [0, L]$, the position of the string. Then the position energy $U$ is the sum of the energy $U_t$ arising from the tension $\mu$ and the energy $U_e$ arising from the external force pressing on the string. We write

$$U = U_t + U_e.$$

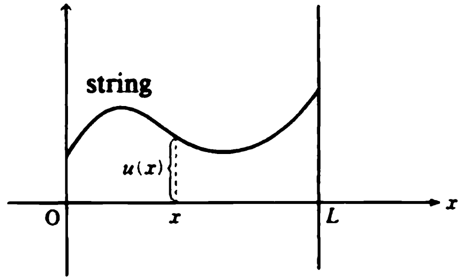



FIGURE 1.5

Here $U_t$ and $U_e$ are given by

$$\begin{aligned} U_t &= \mu\{\text{ length of string } - L\} \\ &= \mu\left\{\int_0^L \sqrt{1+u_x{}^2}\,dx - L\right\} = \mu\int_0^L \left\{\sqrt{1+u_x{}^2} - 1\right\} dx, \\ U_e &= \int_0^L f(x)u(x)\,dx, \end{aligned}$$

respectively, where $u_x := \frac{du}{dx}$. Therefore, the total energy $E$ of the string with position at $u(x)$, $x \in [0, L]$, is given by

$$E(u) = U = \mu\int_0^L \left\{\sqrt{1+u_x{}^2} - 1\right\} dx + \int_0^L f(x)u(x)\,dx.$$

Recall the **least potential energy principle.**

- The equilibrium position should minimize the total energy $E$.

From this principle, we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u+\epsilon v) = 0.$$

Here $u + \epsilon v$ is a position near the equilibrium, and $v$ is an (admissible) function on $[0, L]$. Then we get

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u+\epsilon v) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\mu\int_0^L \left\{\sqrt{1+(u_x+\epsilon v_x)^2} - 1\right\} dx \right. \\ &\qquad\qquad \left. + \int_0^L f(u+\epsilon v)\,dx\right] \\ &= \mu\int_0^L (1+u_x{}^2)^{-1/2} u_x v_x\,dx + \int_0^L fv\,dx \\ &= \int_0^L \left\{-\mu\frac{d}{dx}\left((1+u_x{}^2)^{-1/2}u_x\right) + f\right\} v\,dx \\ &\quad + \left[\mu(1+u_x{}^2)^{-1/2}u_x v\right]_{x=0}^{x=L}, \end{aligned}$$

using the partial integral formula. Thus, we obtain

$$\int_0^L \left\{-\mu \frac{d}{dx}\left((1+u_x^2)^{-1/2}u_x\right)+f\right\} v\,dx \qquad (3.1)$$
$$+\mu\left\{(1+u_x(L)^2)^{-1/2}u_x(L)v(L)-(1+u_x(0)^2)^{-1/2}u_x(0)v(0)\right\}=0.$$

Thus,

(i) if the endpoints of the string being not fixed, then the function $v$ may be chosen arbitrarily, and we get

$$\begin{cases} \mu \frac{d}{dx}\left((1+u_x^2)^{-1/2}u_x\right)=f(x), & \text{on the open interval } (0, L), \\ u_x(0)=u_x(L)=0; \end{cases} \qquad (3.2)$$

(ii) if both the endpoints are fixed, then $v$ should vanish at both of the endpoints $0$ and $L$, and we have

$$\begin{cases} \mu \frac{d}{dx}\left((1+u_x^2)^{-1/2}u_x\right)=f(x), & \text{on the open interval } (0, L), \\ u(0)=\alpha,\ u(L)=\beta. \end{cases} \qquad (3.3)$$

Since

$$\frac{d}{dx}\left((1+u_x^2)^{-1/2}u_x\right)=u_{xx}(1+u_x^2)^{-3/2}, \qquad (3.4)$$

the above nonlinear differential equation can be also written as

$$\mu u_{xx}(1+u_x^2)^{-3/2}=f.$$

In summary, we have

(3.5) RÉSUMÉ. (I) *In order to determine the equilibrium position of a string in the field which the external force acts vertically to the x-axis, one may search for a minimum of the function defined by*

$$E(u):=\mu\int_0^L\left\{\sqrt{1+u_x^2}-1\right\}dx+\int_0^L f(x)u(x)\,dx.$$

(II) *To do the above, one may solve the nonlinear differential equation on the open interval* $(0, L)$ *given by*

$$\mu\frac{d}{dx}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)=f(x) \quad or \quad \mu u_{xx}(1+u_x^2)^{-3/2}=f(x),$$

*under the boundary conditions*

$$u_x(0)=u_x(L)=0 \quad or \quad u(0)=\alpha,\ u(L)=\beta.$$

REMARK. If we take

$$E(u):=\frac{\mu}{2}\int_0^L u_x^2\,dx+\int_0^L f u\,dx,$$

one can derive the **Poisson equation**

$$\mu u_{xx}=f.$$

**3.2. Equation of a vibrating string.** We next consider the equation of a vibrating string. We denote by $u(t, x)$, the position of a string changing with time $t$. The position energy $U(t)$ at $t$ is given by

$$U(t) = \mu \int_0^L \left\{ \sqrt{1 + u_x{}^2} - 1 \right\} dx + \int_0^L f(x)u(x)\, dx.$$

At each point of a string, the kinetic enegy is locally $\frac{1}{2} \cdot \text{mass} \cdot |\text{speed}|^2$, so the kinetic energy $T(t)$ of the string is

$$T(t) = \int_0^L \frac{1}{2} \mu u_t{}^2\, dx.$$

Therefore, the total energy $E(t)$ at time $t$ is given by the formula:

$$\begin{aligned} E(t) &= T(t) - U(t) \\ &= \int_0^L \left\{ \frac{1}{2} \rho u_t{}^2 - \mu \sqrt{1 + u_t{}^2} + \mu - fu \right\} dx. \end{aligned}$$

Here recall **Hamilton's principle**:

- The total energy is defined by

  $$E := \int_{t_1}^{t_2} E(t)\, dt$$

  for an arbitrary position $u(t, x)$ of a string. The real motion of the string from time $t_1$ to time $t_2$ minimizes $E$ among the totality of all possible positions which coincide with the real motion at times $t_1$, $t_2$.

Since $E$ is a function of $u$, we denote it by $E(u)$. For all $v(t, x)$ satisfying

$$v(t_1, x) = v(t_2, x) = 0, \qquad x \in [0, L],$$

the position $u(t, x)$ of the real motion of a string must satisfy

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u + \epsilon v) = 0.$$

Here $E(u)$ is given by

$$E(u) = \int_{t_1}^{t_2} \int_0^L \left\{ \frac{1}{2} \rho u_t{}^2 - \mu \sqrt{1 + u_x{}^2} + \mu - fu \right\} dt\, dx.$$

Therefore, we get

$$\begin{aligned}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} & E(u+\epsilon v) \\ &= \int_{t_1}^{t_2}\int_0^L \left\{\frac{1}{2}\rho\,(u_t+\epsilon v_t)^2 - \mu\left(1+(u_x+\epsilon v_x)^2\right)^{1/2}\right. \\ &\qquad\qquad \left. + \mu - f(u+\epsilon v)\right\} dt\,dx \\ &= \int_{t_1}^{t_2}\int_0^L \left\{\rho\, u_t v_t - \mu\,(1+u_x{}^2)^{-1/2} u_x v_x - f v\right\} dt\,dx.\end{aligned}$$

Using the partial integral formula, we find that it is equal to

$$\begin{aligned}&\int_0^L \left\{\Big[\rho\, u_t v\Big]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \rho\, u_{tt} v\, dt\right\} dx \\ &- \int_{t_1}^{t_2} \left\{\Big[\mu(1+u_x{}^2)^{-1/2} u_x v\Big]_{x=0}^{x=L} - \int_0^L \mu \frac{d}{dx}\left((1+u_x{}^2)^{-1/2} u_x\right) v\, dx\right\} dt \\ &- \int_{t_1}^{t_2}\int_0^L f v\, dt\, dx \\ &= \int_{t_1}^{t_2}\int_0^L \left\{-\rho\, u_{tt} + \mu \frac{d}{dx}\left((1+u_x{}^2)^{-1/2} u_x\right) - f\right\} v\, dt\, dx \\ &- \int_{t_1}^{t_2} \mu \left\{\left(1+u_x(t,L)^2\right)^{-1/2} u_x(t,L) v(t,L)\right. \\ &\qquad \left. - \left(1+u_x(t,0)^2\right)^{-1/2} u_x(t,0) v(t,0)\right\} dt.\end{aligned}$$

Therefore, the last equation should vanish for all $v$ satisfying $v(t_1, x) = v(t_2, x) = 0$, $x \in [0, L]$. Taking first $v$ in such a way that the closure of $\{(t, x); v(t, x) \neq 0\}$ is contained in the set $(t_1, t_2) \times (0, L)$, we get the equation

$$\rho\, u_{tt} - \mu \frac{d}{dx}\left((1+u_x{}^2)^{-1/2} u_x\right) + f = 0 \quad \text{on } (t_1, t_2) \times (0, L). \tag{3.6}$$

Next taking $v$ such that $v(t_1, x) = v(t_2, x) = 0$, $x \in [0, L]$, we get

$$\begin{aligned}\int_{t_1}^{t_2} &\left\{\left(1+u_x(t,L)^2\right)^{-1/2} u_x(t,L) v(t,L)\right. \\ &\qquad \left. - \left(1+u_x(t,0)^2\right)^{-1/2} u_x(t,0) v(t,0)\right\} dt = 0.\end{aligned} \tag{3.7}$$

Hence, we get

(i) The endpoints of a string being not fixed, there are no restrictions of $u$

and $v$, so we have

$$\begin{cases} \rho u_{tt} - \mu \dfrac{d}{dx}\left((1+u_x{}^2)^{-1/2} u_x\right) + f = 0, & \mathbf{R} \times (0, L), \\ u_x(t, L) = u_x(t, 0) = 0, & t \in \mathbf{R}. \end{cases} \tag{3.8}$$

(ii) When the endpoints are fixed, $v(t, L) = v(t, 0) = 0$, Then in this case, (3.7) holds always. Therefore,

$$\begin{cases} \rho u_{tt} - \mu \dfrac{d}{dx}\left((1+u_x{}^2)^{-1/2} u_x\right) + f = 0, & \mathbf{R} \times (0, L), \\ u(t, L) = \alpha,\ u(t, 0) = \beta, & t \in \mathbf{R}. \end{cases} \tag{3.9}$$

Moreover, we need the initial condition at $t = t_0$ in order to determine the motion of a string:

$$\begin{cases} u(t_0, x) = u_0(x), & \\ u_t(t_0, x) = u_1(x), & x \in [0, L]. \end{cases} \tag{3.10}$$

Summing up the above, we have

(3.11) RÉSUMÉ. (I) *To determine the motion of a string under an external force acting vertically the x-axis, for all times* $t_1$, $t_2$, *we may search for the minimizer of the function* $E(u)$ *of* $u$ *defined by*

$$E(u) = \int_{t_1}^{t_2} \left\{\frac{1}{2}\rho u_t{}^2 - \mu\sqrt{1 + u_x{}^2} + \mu - fu\right\} dt\, dx.$$

(II) *The equation of the motion of a string is the following nonlinear wave equation on* $\mathbf{R} \times (0, L)$ *given by*

$$\rho u_{tt} - \mu \frac{d}{dx}\left(\frac{u_x}{\sqrt{1 + u_x{}^2}}\right) + f = 0$$

*or*

$$\rho u_{tt} - \mu u_{xx}(1 + u_x{}^2)^{-3/2} + f = 0.$$

*In order to determine the motion, we may solve this equation under the* **initial condition** *at time* $t = t_0$:

$$u(t_0, x) = u_0(x), \qquad u_t(t_0, x) = u_1(x), \qquad x \in [0, L],$$

*and the* **boundary condition** *at both the end points* $x = 0, L$:

$$u_x(t, L) = u_x(t, 0) = 0 \quad \text{or} \quad u(t, L) = \alpha,\ u(t, 0) = \beta.$$

REMARK. If we take as the function $E$,

$$E(u) := \int_{t_1}^{t_2} \int_0^L \left\{\frac{1}{2}\rho u_t{}^2 - \frac{1}{2}u_x{}^2 - fu\right\} dt\, dx,$$

then we derive the **wave equation**

$$\rho u_{tt} - \mu u_{xx} + f = 0.$$

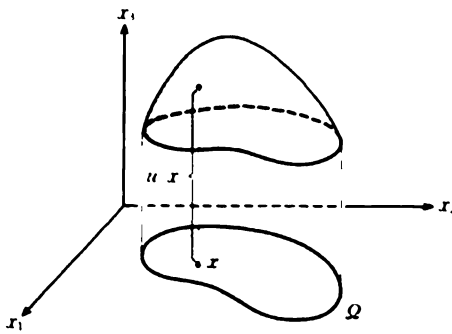



FIGURE 1.6

**3.3. Equation of equilibrium states of membranes.** Assume first that a membrane is at a stationary state under no external force and is put on a domain $\Omega \subset \mathbf{R}^2$. Then when an external force with strength $f(x)$ at $x = (x_1, x_2) \in \Omega$ acts vertically the $xy$-plane $\mathbf{R}^2$, we shall find the equations of equilibrium state and a vibrating motion for the membrane. We sketch them briefly since they are similar to the material in subsections 3.1 and 3.2.

(3.12) We assume the strength of the membrane is constant $\mu$ and its density is also constant $\rho$. Then the total energy of the membrane being at $u(x)$, $x \in \Omega$ is given by

$$E(u) = U_t + U_e,$$

where

$$\begin{aligned} U_t : &= \mu\{\text{surface area of membrane} - |\Omega|\} \\ &= \mu \int_\Omega \left\{\sqrt{1 + |\nabla u|^2} - 1\right\} dx, \end{aligned}$$

and

$$|\nabla u|^2 := \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i}\right)^2, \qquad dx := dx_1\, dx_2, \qquad |\Omega| := \int_\Omega dx$$

$$U_e := \int_\Omega f u\, dx.$$

See Figure 1.6. Therefore ,

$$E(u) = \mu \int_\Omega \left\{\sqrt{1 + |\nabla u|^2} - 1\right\} dx + \int_\Omega f u\, dx.$$

For $v$ a function on $\Omega$ and $u + \epsilon v$ a position of a membrane near the one

$u$ of equilibrium, we get

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u+\epsilon v) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\mu\int_\Omega\left\{\sqrt{1+|\nabla u+\epsilon\nabla v|^2}-1\right\}dx + \int_\Omega f(u+\epsilon v)\,dx\right] \tag{3.13}$$

$$= \mu\int_\Omega\{F\,\langle\nabla u,\nabla v\rangle + fv\}\,dx,$$

where $F := (1+|\nabla u|^2)^{-1/2}$ and $\langle\nabla u,\nabla v\rangle := \sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$. Recall the following Stokes' theorem:

STOKES' THEOREM. *For continuous functions $P$, $Q$ on $\overline{\Omega}$ smooth on $\Omega$,*

$$\int_{\partial\Omega}(P\,dx_1 + Q\,dx_2) = \int_\Omega\left(\frac{\partial Q}{\partial x_1}-\frac{\partial P}{\partial x_2}\right)dx_1dx_2,$$

*where $\overline{\Omega}$ is the closure of a domain $\Omega$.*

Applying Stokes' theorem to $P := -vF\frac{\partial u}{\partial x_2}$, $Q := vF\frac{\partial u}{\partial x_1}$, we get

$$\frac{\partial Q}{\partial x_1}-\frac{\partial P}{\partial x_2} = F\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + v\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(F\frac{\partial u}{\partial x_i}\right).$$

Thus, we get

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u+\epsilon v) = \int_\Omega\left[-\mu\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(F\frac{\partial}{\partial x_i}\right)+f\right]v\,dx + \int_{\partial\Omega}\left\{-vF\frac{\partial u}{\partial x_2}dx_1 + vF\frac{\partial u}{\partial x_1}dx_2\right\}.$$

Here denoting by $d\sigma$ the canonical surface measure of $\partial\Omega$, by $\nu$ the inward unit normal vector at $\partial\Omega$, and by $\frac{\partial u}{\partial\nu}$ the derivative of $u$ in the direction $\nu$, we get

$$\text{the second term of the above} = -\int_{\partial\Omega} vF\frac{\partial u}{\partial\nu}\,d\sigma.$$

Therefore, we obtain

$$\int_\Omega\left[-\mu\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(F\frac{\partial}{\partial x_i}\right)+f\right]v\,dx - \int_{\partial\Omega} vF\frac{\partial u}{\partial\nu}\,d\sigma = 0. \tag{3.14}$$

Thus,

(i) if the membrane is vibrating freely, then $v$ may be taken arbitrarily, and so

$$\begin{cases}\mu\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(F\frac{\partial u}{\partial x_i}\right) = f & \text{in } \Omega,\\[4pt] \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.15}$$

(ii) If the boundary of the membrane is fixed, then since $v = 0$ on $\partial\Omega$, we get

$$\begin{cases} \mu \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(F \frac{\partial u}{\partial x_i}\right) = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

where $\varphi$ is a given boundary value on $\partial\Omega$. Summing up the above, we have

RÉSUMÉ. (I) *To determine the position of the equiblium of a membrane under the external force which acts vertically to the xy-plane* $\mathbf{R}^2$, *search for a minimizer of*

$$E(u) = \mu \int_\Omega \left\{\sqrt{1 + |\nabla u|^2} - 1\right\} dx + \int_\Omega f u \, dx.$$

(II) *To do this, we may solve the nonlinear equation*

$$\mu \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \frac{1}{\sqrt{1 + \sum_{i=1}^{2} (\frac{\partial u}{\partial x_i})^2}} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in } \Omega,$$

*under the boundary condition* $\frac{\partial u}{\partial \nu} = 0$, *on* $\partial\Omega$, *or* $u = \varphi$, *on* $\partial\Omega$.

REMARK 1. If $f = 0$, this equation is called the **minimal surface equation** or the **Euler equation.**

REMARK 2. If we take

$$E(u) := \int_\Omega \left\{\frac{\mu}{2} |\nabla u|^2 - f u\right\} dx,$$

then we get the **Poisson equation**

$$\mu \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} = f \quad (\text{in } \Omega). \tag{3.17}$$

REMARK 3. The nonlinear equation of a vibrating membrane can be obtained in a similar way and is of the form

$$\rho u_{tt} - \mu \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \frac{1}{\sqrt{1 + \sum_{i=1}^{2} (\frac{\partial u}{\partial x_i})^2}} \frac{\partial u}{\partial x_i} \right) + f = 0 \quad (\text{in } \Omega),$$

and the linear wave equation is

$$\rho u_{tt} - \mu \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} + f = 0 \quad \text{in } \Omega.$$

**3.4. Closed geodesics on the standard spheres.** Next we shall consider a more geometric problem. A closed smooth curve in $\mathbf{R}^3$ whose period is $2\pi$ is written of the form

$$\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x)) \in \mathbf{R}^3, \qquad x \in [0, 2\pi].$$

Here periodicity of period $2\pi$ means $\phi(x + 2\pi) = \phi(x)$, i.e., $\phi_i(x + 2\pi) = \phi_i(x)$, $i = 1, 2, 3$. Let us consider the following problem:

**PROBLEM.** *Among such curves, which are the critical points of the* **energy** *given by*

$$E(\phi) := \frac{1}{2}\int_0^{2\pi} \sum_{i=1}^{3} \left(\frac{d\phi_i}{dx}\right)^2 dx? \tag{3.18}$$

To answer this question we take a deformation of such a curve $\phi$, $\phi_\epsilon(x) = (\phi_{\epsilon,1}(x), \phi_{\epsilon,2}(x), \phi_{\epsilon,3}(x))$, $x \in [0, 2\pi]$, where $\phi_0 = \phi$ and $\phi_\epsilon(x + 2\pi) = \phi_\epsilon(x)$, $x \in [0, 2\pi]$. Then

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(\phi_\epsilon) &= \frac{1}{2}\int_0^{2\pi} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{i=1}^{3} \left(\frac{d\phi_{\epsilon,i}(x)}{dx}\right)^2 dx \\
&= \int_0^{2\pi} \sum_{i=1}^{3} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{d\phi_{\epsilon,i}(x)}{dx} \frac{d\phi_i(x)}{dx} dx \\
&= \left[\sum_{i=1}^{3} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,i}(x) \frac{d\phi_i(x)}{dx}\right]_{x=0}^{x=2\pi} \\
&\quad - \int_0^{2\pi} \sum_{i=1}^{3} \left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,i}(x)\right) \frac{d^2\phi_i(x)}{dx^2} dx.
\end{aligned}$$

Since both $\phi_i(x)$ and $\phi_{\epsilon,i}(x)$ are periodic of period $2\pi$, the first term of the final expression above vanishes. Moreover, for

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_\epsilon(x) = \left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,1}(x), \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,2}(x), \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,3}(x)\right),$$

since $\phi_\epsilon$ is an arbitrary deformation of $\phi$, we may take for $\frac{d}{d\epsilon}|_{\epsilon=0}\phi_{\epsilon,i}(x)$, an arbitrary smooth periodic function. Therefore, that the above vanishes implies

$$\frac{d^2\phi_i}{dx^2}(x) = 0, \qquad i = 1, 2, 3.$$

Thus, we get $\phi_i(x) = B_i x + A_i$, $i = 1, 2, 3$, where $A_i$, $B_i$ are constants. But periodicity yields $B_i = 0$; thus, we obtain $\phi_i(x) \equiv A_i$, $x \in [0, 2\pi]$, i.e., we obtain only trivial solutions in this case.

On the other hand, we add to smooth curves $\phi$ of period $2\pi$, the following **constraint condition**: all curves $\phi$ should lie on the unit sphere

$$S^2 := \{(y_1, y_2, y_3) \in \mathbf{R}^3 ; y_1^2 + y_2^2 + y_3^2 = 1\}.$$

We consider the similar problem that among such curves which are the critical points of $E$.

In the same way as before, calculating $\frac{d}{d\epsilon}|_{\epsilon=0}E(\phi_\epsilon) = 0$, after taking a deformation $\phi_\epsilon(x)$, $x \in [0, 2\pi]$, we get

$$\int_0^{2\pi} \sum_{i=1}^{3} \left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\epsilon,i}(x)\right) \frac{d^2\phi_i(x)}{dx^2} dx = 0.$$

Here we must take account of the constraint condition: $\phi_\epsilon(x) \in S^2$, $x \in [0, 2\pi]$.

For this, we consider the tangent space of $S^2$ at $y \in S^2$, which is the plane perpendicular to the vector $y$, i.e.,

$$T_yS^2 := \{V \in \mathbf{R}^3;\ (V, y) = 0\}, \tag{3.19}$$

where $\langle\ ,\ \rangle$ is the standard inner product of $\mathbf{R}^3$. Then each vector $V \in \mathbf{R}^3$ can be decomposed into

$$V = \langle V, y\rangle y + (V - \langle V, y\rangle y), \tag{3.20}$$

and the second term belongs to $T_yS^2$.

Now the constraint condition that $\phi_\epsilon(x) \in S^2$ for each $x$ implies that $\langle \phi_\epsilon(x), \phi_\epsilon(x)\rangle = 1$. Differentiate it at $\epsilon = 0$. Since $\phi_0(x) = \phi(x)$,

$$\left\langle \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \phi_\epsilon(x), \phi(x)\right\rangle = 0, \quad \text{i.e.,}\ \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \phi_\epsilon(x) \in T_{\phi(x)}S^2.$$

At each $y = \phi(x)$, $\frac{d}{d\epsilon}|_{\epsilon=0}\phi_\epsilon(x) \in T_{\phi(x)}S^2$ can be taken to be any element in $T_{\phi(x)}S^2$; by (3.19) the $T_{\phi(x)}S^2$-component of the element

$$\frac{d^2\phi}{dx^2} = \left(\frac{d^2\phi_1}{dx^2}, \frac{d^2\phi_2}{dx^2}, \frac{d^2\phi_3}{dx^2}\right)$$

must vanish. By (3.20), we get

$$\frac{d^2\phi}{dx^2} = \left\langle \frac{d^2\phi(x)}{dx^2}, \phi(x)\right\rangle \phi(x), \tag{3.21}$$

which can be rewritten as

$$\frac{d^2\phi}{dx^2} + \left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx}\right\rangle \phi = 0, \tag{3.22}$$

since by differentiating $\langle\phi(x), \phi(x)\rangle = 1$ at each $x \in [0, 2\pi]$, we get

$$\left\langle \frac{d\phi(x)}{dx}, \phi(x)\right\rangle = 0. \tag{3.23}$$

Differentiating (3.23), we get

$$\left\langle \frac{d^2\phi(x)}{dx^2}, \phi(x)\right\rangle + \left\langle \frac{d\phi(x)}{dx}, \frac{d\phi(x)}{dx}\right\rangle = 0. \tag{3.24}$$

Here due to (3.22), notice that $\langle\frac{d\phi(x)}{dx}, \frac{d\phi(x)}{dx}\rangle$ is constant in $x$. Indeed,

$$\begin{aligned}\frac{d}{dx}\left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx}\right\rangle &= 2\left\langle \frac{d^2\phi}{dx^2}, \frac{d\phi}{dx}\right\rangle \\ &= -2\left\langle \left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx}\right\rangle \phi, \frac{d\phi}{dx}\right\rangle \qquad \text{(by (3.22))} \\ &= -2\left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx}\right\rangle\left\langle \phi, \frac{d\phi}{dx}\right\rangle = 0 \quad \text{(by (3.23)).}\end{aligned}$$

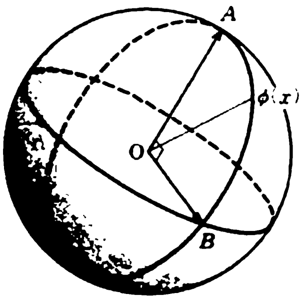



FIGURE 1.7

So assuming $\phi$ is nontrivial, we may set

$$\left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx} \right\rangle = c^2, \qquad c > 0.$$

Then

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx} \right\rangle \phi = 0 &\iff \frac{d^2\phi_i}{dx^2} + c^2\phi_i = 0\,,\ i = 1\,,2\,,3 \\ &\iff \phi_i(x) = A_i \cos cx + B_i \sin cx\,,\ i = 1\,,2\,,3 \\ &\iff \phi(x) = \cos cx\, A + \sin cx\, B\,, \end{aligned}$$

where $A := (A_1, A_2, A_3)$, $B := (B_1, B_2, B_3) \in \mathbf{R}^3$. Necessary and sufficient conditions for such a curve $\phi(x)$, $x \in [0, 2\pi]$, to lie in $S^2$ and to be periodic in $x$ with period $2\pi$ are

$$\begin{cases} (A, A) = (B, B) = 1\,,\ (A, B) = 0\,, & \text{and} \\ c = m\,, & \text{integer.} \end{cases} \tag{3.25}$$

Thus, such a $\phi(x)$ is a great circle of $S^2$ and turns $m$ times when $x$ varies from 0 to $2\pi$ (if $m < 0$, it turns in the reverse direction). See Figure 1.7.

(3.26) RÉSUMÉ. *Among the set of all smooth periodic curves* $\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x)$, $x \in [0, 2\pi]$ *of period* $2\pi$, *all critical points of* $E$ *given by*

$$E(\phi) = \frac{1}{2}\int_0^{2\pi} \sum_{i=1}^{3} \left(\frac{d\phi_i}{dx}\right)^2 dx\,,$$

*are, under the constraint condition that* $\phi$ *lies in the unit sphere*

$$S^2 = \{(y_1, y_2, y_3)\,;\, y_1^2 + y_2^2 + y_3^2 = 1\}\,,$$

*solutions of the differential equation*

$$\frac{d^2\phi}{dx^2} + \left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx} \right\rangle \phi = 0.$$

*All such solutions are great circles of* $S^2$ *turning around* $m$ *times when* $x$ *varies from* $0$ *to* $2\pi$.

## §4. A guide to the further study of the calculus of variations

Summing up the calculations of §3, we shall give some guidance to the further study of the calculus of variations and the benefits of this book.

(1) It would be a misunderstanding of the calculus in §3 if someone were to guess that he found the Euler-Lagrange equation, the only remaining thing might be to solve it, and even if he could not do it, he might find any approximate solution by using a computer. For such a person, the instability theorem stated in §2 of Chapter 5 might be a good moral. It says that any nonconstant harmonic mapping from the unit sphere of dimension higher than two is unstable. The notion of harmonic mapping is quite natural, and it is easy to find the Euler-Lagrange equation. Nevertheless, this theorem says one can only constant mappings even to intend to find an approximate solution unless one comes up with ideas. But to the contrary, Theorem (2.52) in Chapter 6 claims that any smooth mapping of the unit sphere of lower dimension than eight into itself can be deformed to a harmonic mapping. We should continue to elaborate to solve the equation.

(2) Following the next chapter, we shall treat the notion of "*a manifold*" and in addition, "*an infinite dimensional manifold*". Our only aim is to derive and solve the Euler-Lagrange equation. This seems to be a long detour; however, the reason for using the notion of manifold is this: Once one seizes the concepts of *a manifold, a smooth function on it, its tangent vectors*, then one can see the calculations of §3 in perspective. This is the most important thing in global differential geometry.

The meaning of the calculations in §3 is this: We took

$$\begin{aligned} X &= \text{the totality of all possible positions of a string} \\ &= \text{the totality of smooth functions on } [0, L], \end{aligned}$$

for the problem of determining the equilibrium of a string in subsection 3.1, and we took $X =$ the totality of smooth periodic mappings of period $2\pi$ of $[0, 2\pi]$ into $S^2$, for the problem of finding closed geodesics in the unit sphere in subsection 3.4. We then take a deformation $u + \epsilon v$ or $\phi_\epsilon$ of $u \in X$, $\phi \in X$, respectively, and we require the condition

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(u + \epsilon v) = 0 \quad \text{or} \quad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(\phi_\epsilon) = 0,$$

for all deformations. At first glance, this might appear rather strange, but it is quite natural if one introduces the notion of *manifold*:

Let us consider a manifold $X$, and a differentiable function $E$ on $X$. This guarantees we can consider smooth curves in $X$, a differentiation of $E$, etc. In this way, the meaning of the rather lengthy calculation in §3 is

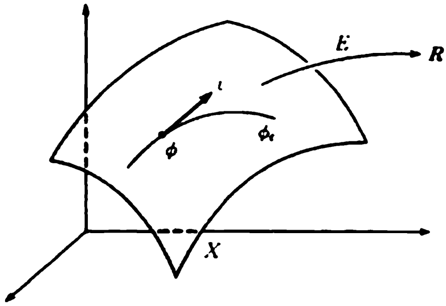



FIGURE 1.8

clearly like the situation of a surface in 3-dimensional Euclidean space. See Figure 1.8.

- Take a deformation of $\phi_\epsilon$ of $\phi$, that is, a smooth curve $\epsilon \mapsto \phi_\epsilon$ in $X$.

- Then consider the tangent vector, say $v$, of this smooth curve:
$$v = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_\epsilon .$$

- We see $\frac{d}{d\epsilon}|_{\epsilon=0}E(\phi_\epsilon)$ is the *partial derivative* of $E$ at $\phi$ in the direction $v$:
$$dE_\phi(v) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(\phi_\epsilon).$$

- Therefore, $\phi$ *a critical point* of $E$ means that all the derivatives of $E$ at $\phi$ in the directions $v$ vanish, i.e., the *total derivative* of $E$ at $\phi$ vanishes:
$$dE_\phi = 0 \iff dE_\phi(v) = 0 \quad \text{for each } v.$$

Thus, that the first derivative of $E$ at $\phi$ is zero is just that $\phi$ is a critical point of a function $E$ on $X$. This enables us to understand the Palais-Smales condition (C) which is a key to the theory of the calculus of variations.

Even if it is difficult to solve problems of the calculus of variations, it is very important to imagine a 3-dimensional geometrical figure. The concept of *a manifold* in a differential geometry is to guarantee this magic. Of course, one needs some effort to get such magic.

**Exercises**

1.1. For a smooth function $u$ on $[0, L]$, we put
$$J(u) := \int_0^L {u_x}^p \, dx \quad (0 < p < \infty), \qquad E(u) := \int_0^L {u_x}^2 \, dx,$$

where $u_x = u'(x) = \frac{du}{dx}$.

(i) Find the Euler-Lagrange equation corresponding to $E$.

(ii) Find the Euler-Lagrange equation corresponding to $J$.

(iii) For $0 < \epsilon < \infty$, find the Euler-Lagrange equation to $E_\epsilon(u) := E(u) + \epsilon J(u)$.

1.2. Find a general solution of the following ordinary equation for an unknown function $y = y(x)$,

$$a \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = f(x).$$

Here $a$ is a positive constant, $f(x)$ is a given function, and $y' = \frac{dy}{dx}$.

1.3. Let $\Omega \subset \mathbf{R}^2$, be a bounded domain, and let

$$\Delta = -\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial}{\partial x_2{}^2}\right), \; x = (x_1, x_2).$$

Then using the method of the separation of variables, solve the following differential equation

$$\rho u_{tt} + \mu \Delta u = 0 \quad \text{on } \mathbf{R} \times \Omega,$$

under the initial condition at $t = t_0$,

$$u(t_o, x) = u_0(x), \qquad u_t(t_0, x) = u_1(x),$$

and the boundary condition on $\partial\Omega$,

$$u(t, x) = 0, \qquad (t, x) \in \mathbf{R} \times \partial\Omega.$$

1.4. Let $(x, y) = (x_1, x_2) \in \mathbf{R}^2$. Then if $u = u(x, y)$, the equation of a minimal surface is

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x{}^2+u_y{}^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x{}^2+u_y{}^2}}\right) = 0.$$

Show that this is equivalent to the equation

$$u_{xx}(1+u_y{}^2) - 2u_{xy}u_x u_y + u_{yy}(1+u_x{}^2) = 0.$$

**≪ Coffee Break ≫ Classical mechanics**

In his epochmaking book *Principia mathematica philosophiae naturalis* (1687), I. Newton clarified the motions of planets of the solar system which obey the following three laws:

The first law ⋯ the law of inertia,
the second law ⋯ the law of motion,
the third law ⋯ the law of action and reaction.

Here we note briefly the second law "the law of motion" and its development.

By the second law, the orbit of a particle of mass $m$ in the space $\mathbf{R}^3$ with a potential field given by a function $V(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, traces a curve $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ which obeys the equation

$$m\ddot{\mathbf{x}} + \operatorname{grad} V(\mathbf{x}) = 0. \tag{1}$$

Here we denote the first and second derivatives of $\mathbf{x}$ with respect to time $t$ by $\dot{\mathbf{x}} = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt})$ and $\ddot{\mathbf{x}} = (\frac{d^2x_1}{dt^2}, \frac{d^2x_2}{dt^2}, \frac{d^2x_3}{dt^2})$. By definition,

$$\operatorname{grad} V(\mathbf{x}) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right),$$

so (1) can be written as

$$\frac{d}{dt}(m\dot{x}_i) + \frac{\partial V}{\partial x_i} = 0, \qquad i = 1, 2, 3. \tag{1$'$}$$

Moreover, putting $\mathbf{F} := -\operatorname{grad} V(\mathbf{x})$, called a field of conservation potential, (1) is also written as

$$\mathbf{F} = m\ddot{\mathbf{x}}, \tag{1$''$}$$

which is the famous formula in classical physics.

Euler and Lagrange reformulated this into the following form (which is the origin of the method of variations).

Consider a function on $\mathbf{R}^3 \times \mathbf{R}^3$, called a **Lagrange function**, given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) := \frac{m}{2}\|\dot{\mathbf{x}}\|^2 - V(\mathbf{x}), \qquad (\mathbf{x}, \dot{\mathbf{x}}) \in \mathbf{R}^3 \times \mathbf{R}^3.$$

Here we regard $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbf{R}^3$ as an independent variable. Then we consider the equation of motion of a particle through two points $\mathbf{x}_0$, $\mathbf{x}_1$ at times $t_0$, $t_1$ $(t_0 < t_1)$. Euler and Lagrange showed the orbit of motion of a particle is given by a curve which among the curves $\sigma(t) = \mathbf{x}(t)$ satisfying

$$\sigma(t_0) = \mathbf{x}_0, \qquad \sigma(t_1) = \mathbf{x}_1,$$

minimizes the following function $E$: for any curve in $\mathbf{R}^3$,

$$[t_0, t_1] \ni t \mapsto \sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t)) \in \mathbf{R}^3,$$

we define

$$E(\sigma) := \int_{t_0}^{t_1} L(\sigma(t), \dot{\sigma}(t))\, dt. \tag{3}$$

Indeed, the orbit of the motion, $\sigma = \sigma(t)$ satisfies the condition that for all $\mathbf{R}^3$-valued functions $h(t) = (h_1(t), h_2(t), h_3(t)) \in \mathbf{R}^3$ on $[t_0, t_1]$ satisfying $h(t_0) = h(t_1) = 0$,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(\sigma + \epsilon h) = 0. \tag{4}$$

Using the partial integral formula, we get

$$\begin{aligned}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(\sigma+\epsilon h) &= \int_{t_0}^{t_1} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(\sigma(t)+\epsilon h(t), \dot\sigma(t)+\epsilon \dot h(t))\,dt \\ &= \int_{t_0}^{t_1} \sum_{i=1}^{3} \left\{ \frac{\partial L}{\partial x_i} h_i + \frac{\partial L}{\partial \dot x_i} \dot h_i \right\} dt \\ &= \int_{t_0}^{t_1} \sum_{i=1}^{3} \left\{ \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_i} \right\} h_i\,dt + \left[ \sum_{i=1}^{3} \frac{\partial L}{\partial \dot x_i} h_i \right]_{t=t_0}^{t=t_1} \\ &= \int_{t_0}^{t_1} \sum_{i=1}^{3} \left\{ \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_i} \right\} h_i\,dt.\end{aligned}$$

Here $h = (h_1, h_2, h_3)$ is any smooth function with $h(t_0) = h(t_1) = 0$. Therefore, the equation which the curve $\sigma(t) = \mathbf{x}(t)$ satisfying (4) should obey is

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_i} = 0, \qquad i = 1, 2, 3, \tag{5}$$

which is called the Euler-Lagrange equation of motion. Moreover, since the function $L$ is given by (2),

$$\frac{\partial L}{\partial \dot x_j} = m\dot x_j, \qquad \frac{\partial L}{\partial x_j} = -\frac{\partial V}{\partial x_j}, \qquad j = 1, 2, 3,$$

which are substituted into (5), and (1′) is obtained.

As above, Euler and Lagrange reconstructed Newton's motion equation through the method of variations.

After them, W.R. Hamilton called $\mathbf{p} := m\dot{\mathbf{x}}$, the momentum, and considered the function $H$ on $\mathbf{R}^3 \times \mathbf{R}^3$ (called the **Hamilton function**)

$$H(\mathbf{x}, \mathbf{p}) := \frac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{x}), \qquad (\mathbf{x}, \mathbf{p}) = (x_1, x_2, x_3, p_1, p_2, p_3) \in \mathbf{R}^3 \times \mathbf{R}^3, \tag{6}$$

and the equation for a curve $(\mathbf{x}(t), \mathbf{p}(t)) = (x_1(t), x_2(t), x_3(t), p_1(t), p_2(t), p_3(t))$ in $\mathbf{R}^3 \times \mathbf{R}^3$ in time $t$,

$$\begin{cases} \dot x_i = \dfrac{\partial H}{\partial p_i}, \\ \dot p_i = -\dfrac{\partial H}{\partial x_i}, \qquad i = 1, 2, 3, \end{cases} \tag{7}$$

which is called **Hamilton's canonical equation**. The first term of (6), and the second term of (6) are called the kinetic energy, the potential energy, respectively. Since the function $H$ is given by (6),

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \qquad \frac{\partial H}{\partial x_i} = \frac{\partial V}{\partial x_i}, \qquad i = 1, 2, 3,$$

which is substituted into (7), then $(1')$ is obtained. Therefore, Newton's equation of motion can be obtained through Hamilton's canonical equation corresponding to Hamilton's function on $\mathbf{R}^3 \times \mathbf{R}^3$.

The theory for studying the Euler-Lagrange motion equation (5) is called **analytical dynamics**. The theory of Hamilton's canonical equation can be formulated into a geometry on the cotangent bundle $T^*M$ of a manifold $M$, which is one of most important theories of differential geometry and has been developed greatly. This beautiful theory is called **classical mechanics**. We recommend for the mathematical theory of classical mechanics, a famous book by V. I. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Math., vol. 60, Springer-Verlag, Berlin and New York, 1978. Classical mechanics is the foundation of both mathematics and physics and is a treasureland of unsolved problems.

In 1744, in Berlin, L. Euler while 37 years old, wrote the first book in history about methods of variation. He moved to Petersburg, in Russia, at the age of 59. He lost his eyesight, but remained active mathematically until his death at the age of 76. He is called a father of Russian Mathematics.

CHAPTER 2

# Manifolds

The notion of *differentiation* initiated by Newton, Leibnitz, Euler, and Lagrange needed to be extended to differentiation on an infinite dimensional space which goes beyond ordinary differentiation or the partial derivative with respect to $n$ variables. We start to lay the foundation to carry this out. To do this, we shall need the notion of a *manifold.*

From the partial derivatives of $n$ variables, one got naturally the notion of an $n$-dimensional manifold, which gave a foundation for Einstein's general relativity and for Riemannian geometry. But it was not sufficient to deal in earnest with the calculus of variations. An infinite dimensional manifold was necessary. From the late 1950's to the mid 1960's, this theory was established. We shall explain it in detail.

## §1. Continuity, differentiation, and integration

We start with continuity and differentiation. We shall extend these notions given in the first undergraduate course, to a Banach space. For readers not familiar with Banach spaces, not much is lost if it is regarded as the calculus on the $n$-dimensional Euclidean space $\mathbf{R}^n$.

### 1.1. Continuity and Linear Operators.

(1.1) *Banach spaces.* A **Banach space** is by definition $(E, \| \ \|)$ such that $E$ is a vector space over $\mathbf{R}$ with the following three conditions (i), (ii), (iii) satisfied:

(i) Addition and scalar multiplication.

$$x, y \in E, \ \lambda, \mu \in \mathbf{R} \Longrightarrow \lambda x + \mu y \in E,$$

are defined.

(ii) A **norm** $\| \ \|$ on $E$, i.e.,

(ii-a) $\|x + y\| \le \|x\| + \|y\|$, $x, y \in E$,

(ii-b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbf{R}$, $x \in E$,

(ii-c) $\|x\| \ge 0$, $x \in E$, and equality holds if and only if $x = 0$, is given and satisfies the following condition:

(iii) Completeness.

A sequence $\{x_n\}_{n=1}^{\infty}$ in $E$ is a **Cauchy sequence** if $\|x_n - x_m\| \to 0$ (if $n, m \to \infty$). A vector space $(E, \| \|)$ with a norm $\| \|$ is **complete** if for any Cauchy sequence $\{x_n\}_{n=1}^{\infty}$, there exists a point $p$ in $E$ such that $x_n$ converges to $p$, i.e., $\|x_n - p\| \to 0$ if $n \to \infty$. (We denote this simply by $x_n \to p$.) We assume this condition.

(1.2) *A Hilbert space.* A Hilbert space is by definition, $(H, \langle , \rangle)$ such that

(i) $H$ is a vector space over $\mathbf{R}$,
(ii) an **inner product** $\langle , \rangle$ on $H$, i.e.,
(ii-a) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$, $\lambda, \mu \in \mathbf{R}$, $x, y, z \in H$,
(ii-b) $\langle x, y\rangle = \langle y, x\rangle$, $x, y \in H$,
(ii-c) $\langle x, x\rangle \geq 0$, $x \in H$, equality holds if and only if $x = 0$, is given and if we give a norm on $H$ by $\|x\| := \langle x, x\rangle^{1/2}$, $x \in H$,
(iii) $(H, \| \|)$ is complete in the sense of (1.1).

EXAMPLE 1. Let $H = \mathbf{R}^n$. We define the usual inner product on $H$ by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \quad x = (x_1, \dots, x_n), \; y = (y_1, \dots, y_n) \in \mathbf{R}^n.$$

Then $(H, \langle \; , \; \rangle)$ is an $n$-dimensional Hilbert space, which is called $n$-dimensional **Euclidean space**.

EXAMPLE 2. (i) Let $0 < p < \infty$. We denote by $E = L_p(\mathbf{R}^n)$ the set of all real valued measurable functions $f$ on $\mathbf{R}^n$ satisfying

$$\|f\|_p := \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x)|^p \, dx_1 \cdots dx_n\right)^{1/p} < \infty.$$

For $f_1, f_2 \in L_p(\mathbf{R}^p)$, we write $f_1 = f_2$ if $f_1(x) = f_2(x)$ for almost every $x \in \mathbf{R}^n$, and we set

$$(\lambda f_1 + \mu f_2)(x) = \lambda f_1(x) + \mu f_2(x), \qquad x \in M.$$

Then $(E, \| \|_p)$ is a Banach space. If $p = 2$,

$$\langle f_1, f_2\rangle := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(x) f_2(x) \, dx_1 \cdots dx_n,$$

gives an inner product $\langle , \rangle$ on $L_2(\mathbf{R}^n)$, and $(L_2(\mathbf{R}^n), \langle , \rangle)$ is a Hilbert space.

(ii) Let $C^0([0, 1])$ be the space of all real-valued continuous functions on the closed interval $[0, 1]$ with the same addition and scalar multiplication as the above example (i). Then by the following norm $\| \|_\infty$, it is a Banach space:

$$\|f\|_\infty := \sup\{|f(x)|; x \in [0, 1]\}.$$

In the following, we denote Banach spaces by, $E, F, \dots$, and denote by the same letter $\| \|$, each norm on $E, F, \dots$.

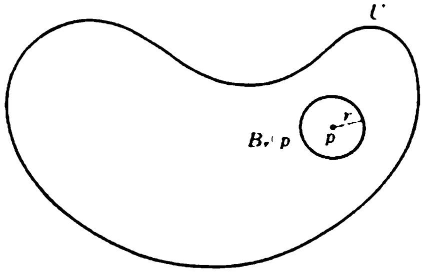



FIGURE 2.1

For $p \in E$, $r > 0$, the **open ball** centered at $p$ and with radius $r$ is the set

$$B_r(p) := \{x \in E;\ \|x - p\| < r\}.$$

A subset $U$ of $E$ is called an **open set** if for each point $p$ in $U$, we can take a small number $r > 0$ such that $B_r(p) \subset U$. See Figure 2.1.

(1.3) *Continuous functions.* Let $E$, $F$ be two Banach spaces and let $U$ be an open subset of $E$. An $F$-valued function $f$ defined on $U$, $f : U \to F$, is **continuous** at $p \in U$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $U$ convergent to $p$, $f(x_n) \to f(p)\ (n \to \infty)$. A function that is continuous at each point in $U$ is said to be continuous on $U$.

A mapping $T : E \to F$ is said to be linear if $T$ satisfies the equation

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y), \qquad \lambda, \mu \in \mathbf{R},\ x, y \in E.$$

A mapping $T$ is **bounded** if there exists a positive constant $C > 0$ such that

$$\|T(x)\| \le C\|x\| \quad \text{for all} \quad x \in E. \tag{1.4}$$

(1.5) Any bounded linear mapping is continuous, and conversely any continuous linear mapping is bounded.

In fact, if a linear mapping $T : E \to F$ is bounded, since

$$\|T(x_n) - T(p)\| = \|T(x_n - p)\| \le C\|x_n - p\| \to 0 \quad (n \to \infty),$$

$T$ is continuous. Conversely, assume that a linear mapping $T$ is continuous at $p \in E$. Then there exists $B > 0$ such that

$$\|x - p\| \le B \Longrightarrow \|T(x - p)\| = \|T(x) - T(p)\| \le 1.$$

Then if $\|y\| \le B$, $\|T(y)\| \le 1$. Hence, for all $0 \ne z \in E$,

$$\|T(z)\| = \frac{\|z\|}{B}\left\|T\left(\frac{B}{\|z\|}z\right)\right\| \le \frac{\|z\|}{B}.$$

Therefore, taking $C = \frac{1}{B}$, we get the inequality (1.4). □

(1.6) We denote by $L(E, F)$ the totality of all bounded linear mappings of $E$ into $F$. Then $L(E, F)$ becomes a Banach space as follows:

(i) Addition and scalar multiplication. For $T_1, T_2 \in L(E, F)$, $\lambda, \mu \in \mathbf{R}$,

$$(\lambda T_1 + \mu T_2)(x) := \lambda T_1(x) + \mu T_2(x), \qquad x \in E.$$

(ii) Norm on $L(E, F)$. For $T \in L(E, F)$, we define its norm $\|T\|$ to be the infimum of the constants $C$ satisfying (1.4), i.e.,

$$\|T\| := \sup\{\|T(x)\|/\|x\|;\ 0 \neq x \in E\}.$$

Then by definition,

$$\|T(x)\| \leq \|T\|\,\|x\|, \qquad x \in E.$$

(iii) Completeness of $(L(E, F), \|\ \|)$. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L(E, F)$ with respect to $\|\ \|$. Then for each $x \in E$, $\{T_n(x)\}_{n=1}^{\infty}$ is also Cauchy sequence in $F$. Since $F$ is complete, there exists $y$, say $T(x)$, such that $T_n(x) \to y \in F$. We need only check such $T$ belongs to $L(E, F)$ and $\|T_n - T\| \to 0$ (see exercise 2.2).

(1.7) *Direct product of Banach spaces.* For $n$ Banach spaces $E_1, \ldots, E_n$, let

$$E := E_1 \times \cdots \times E_n = \{(x_1, \ldots, x_n);\ x_1 \in E_1, \ldots, x_n \in E_n\}.$$

Define the addition and scalar multiplication on $E$ by,

$$\lambda x + \mu y = (\lambda x_1 + \mu y_1, \ldots, \lambda x_n + \mu y_n),$$

for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in E$, $\lambda, \mu \in \mathbf{R}$, and define the norm $\|x\|$ of $x = (x_1, \ldots, x_n) \in E$ by

$$\|x\| = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2}.$$

Then $(E, \|\ \|) = (E_1 \times \cdots \times E_n, \|\ \|)$ becomes a Banach space, called a **direct product Banach space.**

(1.8) *n-multilinear mappings.* A mapping $T$ of a direct product Banach space $E_1 \times \cdots \times E_n$ into a Banach space $F$ is called ***n*-multilinear** if for each number $i = 1, \ldots, n$,

$$\begin{aligned} &T(x_1, \ldots, \lambda x_i + \mu x_i', \ldots, x_n) \\ &\quad = \lambda T(x_1, \ldots, x_i, \ldots, x_n) + \mu T(x_1, \ldots, x_i', \ldots, x_n) \end{aligned}$$

for $\lambda, \mu \in \mathbf{R}$, $x_i, x_i' \in E_i$. We denote by $L(E_1, \ldots, E_n; F)$, the totality of all bounded $n$-multilinear mappings of $E_1 \times \cdots \times E_n$ into $F$. Here an $n$-multilinear mapping $T : E_1 \times \cdots \times E_n \to F$ is **bounded** if there exists a positive constant $C$ such that

$$\|T(x_1, \ldots, x_n)\| \leq C\,\|x_1\| \cdots \|x_n\|, \qquad x_i \in E_i.$$

The space $L(E_1, \ldots, E_n; F)$ becomes a Banach space as follows. The boundedness of $T$ is equivalent to its continuity in the same way as (1.5).

The addition and scalar multiplication of $L(E_1, \dots, E_n; F)$ is clearly defined. The norm is defined as follows. For $T \in L(E_1, \dots, E_n; F)$, define $\|T\|$ as the infimum of the above constants $C$, i.e.,

$$\|T\| := \sup\{\|T(x_1, \cdots, x_n)\|/\|x_1\| \cdots \|x_n\|;\ 0 \neq x_i \in E_i,\ 1 \leq i \leq n\}.$$

(1.9) In particular, if $E_1 = \cdots = E_n = E$, we denote $L(E_1, \cdots, E_n; F) = L^n(E; F)$. There exists a canonical isomorphism (as Banach spaces):

$$L(E_1, E_2; F) \cong L(E_1, L(E_2, F)).$$

In fact, the isomorphism is given as follows: For $T \in L(E_1, E_2; F)$, define a linear mapping of $E_1$ into $L(E_2, F)$, $E_1 \ni x_1 \mapsto T(x_1) \in L(E_2, F)$, by

$$T(x_1)(x_2) := T(x_1, x_2), \quad x_2 \in E_2.$$

Repeating the above inductively, we get

$$L(E_1, \dots, E_n; F) \cong L(E_1, \dots, E_i; L(E_{i+1}, \dots, E_n; F)), \tag{1.10}$$

in particular,

$$L^{n+1}(E; F) \cong L(E; L^n(E; F)). \tag{1.10$'$}$$

**1.2. Differentiation.** Here we consider a mapping $f : U \to F$, not linear in general, defined on an open subset $U$ of $E$. $f$ is **differentiable** at $p \in U$ if there exists a bounded linear mapping $T \in L(E, F)$ such that

$$\|f(p + x) - f(p) - T(x)\|/\|x\| \to 0 \qquad \text{as } x \to 0. \tag{1.11}$$

Note that since $U$ is open in $E$, if $x$ is close to $0$, then $p + x \in U$ and $f(p + x)$ is well defined.

This $T \in L(E, F)$ is uniquely determined as we shall show below, thus, $T$ is called a **differential** of $f$ at $p$, denoted by $df_p$, $f_{*p}$, $f'(p)$.

In fact, assume that there is another $T' \in L(E, F)$, then

$$\begin{aligned} T(x) - T'(x) = & -\{f(p + x) - f(p) - T(x)\} \\ & + \{f(p + x) - f(p) - T'(x)\}, \end{aligned}$$

and using (1.11) we get

$$\|T(x) - T'(x)\|/\|x\| \to 0, \qquad \text{as } x \to 0. \tag{1.11$'$}$$

Then for arbitrarily given $0 \neq z \in E$, we shall show $\|T(z) - T'(z)\| = 0$. For a sufficiently small $\epsilon > 0$, we get $p + \epsilon z \in U$, and so substituting $x = \epsilon z$ into (1.11′) we get

$$\|T(\epsilon z) - T'(\epsilon z)\|/\|\epsilon z\| \to 0, \qquad \text{as } \epsilon \to 0.$$

On the other hand, since $T$, $T'$ are linear, the left hand side coincides with $\|T(z) - T'(z)\|/\|z\|$ which is independent of $\epsilon$. Therefore, we obtain $\|T(z) - T'(z)\| = 0$, which is the desired result. See Figure 2.2, next page. □

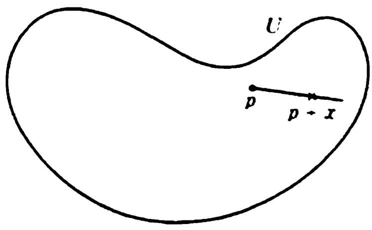



FIGURE 2.2

(1.12) If $f : U \to F$ is differentiable at $p$, then its differential $df_p \in L(E, F)$ is defined by setting

$$df_p(x) = \lim_{t\to} \frac{1}{t}\{f(p+tx) - f(p)\} \quad \text{for } x \in E.$$

Here the right-hand side is a limit in the Banach space $F$, and we denote it also by

$$\left.\frac{d}{dt}\right|_{t=0} f(p+tx),$$

which we call the $x$-direction derivative of $f$ at $p$.

Conversely, we have the following

PROPOSITION (Zorn [Z]). *Assume that $f : U \to F$ satisfies the condition that for each $p \in U$,*

$$df_p(x) := \left.\frac{d}{dt}\right|_{t=0} f(p+tx), \qquad x \in E$$

*exists and $df_p \in L(E, F)$, i.e., $E \ni x \mapsto df_p(x) \in F$ is bounded linear. Moreover, we assume $E \ni p \mapsto df_p \in L(E, F)$ is continuous. Then $f$ is differentiable on $U$ and $df_p$ is a differential of $f$ at $p$.*

OUTLINE OF PROOF. We need the mean value theorem (1.26) below, which asserts that taking $v \in E$ such that $p + tv \in U$, $0 \le t \le 1$,

$$f(p+v) - f(p) = \int_0^1 df_{p+tv}(v)\,dt.$$

For such $p$, $v$, we put

$$\omega(p, v) = f(p+v) - f(p) - df_p(v) \in F. \qquad (1.12')$$

Then by a proof of the Hahn-Banach theorem below (1.27), there exists a bounded linear mapping $\lambda : F \to \mathbf{R}$ such that

$$\lambda(\omega(p, v)) = \|\omega(p, v)\| \quad \text{and} \quad \|\lambda\| = 1.$$

Then by (1.12′),

$$\begin{aligned}\|\omega(p, v)\| &= \lambda(\omega(p, v)) \\ &= \lambda(f(p+v) - f(p)) - \lambda(df_p(v)) \\ &= \int_0^1 \lambda((df_{p+tv} - df_p)v)\,dt.\end{aligned}$$

Together with $\|\lambda\| = 1$, we get

$$\|\omega(p, v)\| \leq \|v\| \sup_{0\leq t\leq 1} \|df_{p+tv} - df_p\|.$$

By the assumption that $df_p$ is continuous in $p$, we have

$$\sup_{0\leq t\leq 1} \|df_{p+tv} - df_p\| \to 0 \quad \text{as } \|v\| \to 0.$$

Together with the above, we get

$$\lim_{v\to 0} \|\omega(p, v)\|/\|v\| = 0,$$

which implies that $f$ is differentiable and its differential at $p$ coincides with $df_p$. □

The interpretation of the above Zorn's proposition is that a function $f$ which is partially differentiable in all directions and satisfies the additional continuity assumptions about the partial differentials, is in fact, differentiable.

(1.13) If $f$ is differentiable at $p$, then $f$ is continuous at $p$.

Indeed, if $x \to 0$, the assertion follows from the inequality

$$\|f(p+x) - f(p)\| \leq \|T(x)\| + \|f(p+x) - f(p) - T(x)\| \to 0. \quad \square$$

The following facts can be seen by definition, their proofs are left to the readers.

(1.14) If $f$ is differentiable at $p$, and $p \in V \subset U$ open subsets, then the restriction $g = f|_V$, of $f$ to $V$ is differentiable at $p$ and $df_p = dg_p$.

(1.15) If $f : U \to F$ is constant, then $f$ is differentiable at any $p$ and $f_p = 0$.

(1.16) If $T : E \to F$ is a bounded linear mapping, $p \in U \subset E$, an open set, and $f = T|_U$ the restriction to $U$, then $f$ is differentiable at $p$ and $df_p = T$.

(1.17) If $f$, $g : U \to F$ are differentiable at $p \in U$, and $\lambda$, $\mu \in \mathbf{R}$, then $\lambda f + \mu g$ is also differentiable at $p$ and $d(\lambda f + \mu g)_p = \lambda df_p + \mu dg_p$.

(1.18) *Diffferentiation law of the composition.* Let $E$, $F$, $G$ be three Banach spaces, let $U \subset E$, $V \subset F$ be open sets, let $f : U \to F$, $g : V \to G$ be mappings differentiable at $p \in U$, $f(p) \in V$, respectively, and assume $f(U) \subset V$. Then the composition $g \circ f : U \to G$ is differentiable at $p$ and $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

EXAMPLE 3. A mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ can be expressed using the coordinates of $\mathbf{R}^m$, $\mathbf{R}^n$ as

$$f(x) = {}^t(f_1(x), \dots, f_n(x)), \qquad x = {}^t(x_1, \dots, x_m) \in U \subset \mathbf{R}^m,$$

where ${}^t$ is the transposed matrix. Then the differentiation $df_p : \mathbf{R}^m \to \mathbf{R}^n$ of $f$ at $p \in U$ is given by

$$(df_p)(u) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(p) & \cdots & \frac{\partial f_n}{\partial x_m}(p) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbf{R}^m.$$

In fact, if we denote by $T$ the $(n, m)$-matrix on the right-hand side of the above, then the $i$th component of $f(p+u)-f(p)-T(u)$ coincides with

$$f_i(p+u) - f_i(p) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}(p)\, u_j.$$

Therefore, if each $f_i$ is differentiable at $p$, then

$$\|f(p+u) - f(p) - T(u)\|/\|u\| \to 0 \quad \text{as } u \to 0. \qquad \square$$

(1.19) *$C^1$-function.* If $f : U \to F$ is differentiable everywhere on $U$, then $df : p \mapsto df_p$ is an $L(E, F)$-valued function on $U$. We say $f$ is $\mathbf{C}^1$ if $df : U \to L(E, F)$ is continuous.

Moreover, if $df$ is differentiable at $p \in U$, the differentiation of $U \ni x \mapsto df_x \in L(E, F)$ at $p$ is $d(df)_p \in L(E, L(E, F))$ and $T = d(df)_p$ satisfies

$$\|(df)_{p+x} - df_p - T(x)\|/\|x\| \to 0 \quad \text{as } x \to 0.$$

We write also $d^2 f_p$ for $d(df)_p$ and call it **second derivative**. Using (1.9) and the identification $L(E, L(E, F)) \cong L^2(E; F)$, we find that $d^2 f$ is a bilinear mapping of $E \times E$ into $F$ and moreover,

(1.20) $d^2 f_p$ is symmetric, i.e.,

$$d^2 f_p(x, y) = d^2 f_p(y, x), \qquad x, y \in E.$$

This implies that twice differentiation of $f$ at $p$, first in the $x$-direction and then in the $y$-direction coincides with the twice differentiation first, in the $y$-direction and then in the $x$-direction. This can be proved using the mean value theorem in subsection 1.3, and it is left for the reader (see also (1.21) below).

If $d^2 f : U \ni p \mapsto d^2 f_p$ is continuous, we say $f$ is $\mathbf{C}^2$.

(1.21) *$C^\infty$ functions.* Define inductively, if $d^k f : U \to L^k(E; F)$ is defined and differentiable at $p$, then

$$d^{k+1} f_p = d(d^k f)_p \in L(E; L^k(E; F)) \cong L^{k+1}(E; F)$$

is called $(k+1)$th differentiation.

$$d^{k+1} f_p : \underbrace{E \times \cdots \times E}_{k+1} \to F$$

is a $(k+1)$-tuple linear mapping and is symmetric, i.e., for all permutations $\sigma$ of $\{1, \ldots, k+1\}$,

$$d^{k+1} f_p(x_{\sigma(1)}, \cdots, x_{\sigma(k+1)}) = d^{k+1} f_p(x_1, \cdots, x_{k+1}), \quad x_i \in E.$$

If $d^{k+1} f : U \ni p \mapsto d^{k+1} f_p \in L^{k+1}(E; F)$ is continuous, then $f$ is said to be $C^{k+1}$. If all $k \geq 1$, $f$ is $C^k$ on $U$, then $f$ is said to be $\mathbf{C}^\infty$ on $U$.

If $f$ is $C^k$, then for all $x_1, \ldots, x_k \in E$, it follows that

$$d^k f_p(x_1, \ldots, x_k) = \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\bigg|_{(t_1, \ldots, t_k)=(0, \ldots, 0)} f(p + t_1 x_1 + \cdots + t_k x_k)$$

which is very useful for calculating $d^k f_p$.

For its proof, for instance for $k = 2$ with $x_1, x_2 \in E$, we get

$$\begin{aligned} d^2 f_p(x_1, x_2) &= d_p\left\{\frac{d}{dt}\bigg|_{t=0} f(p + tx_1)\right\}(x_2) \\ &= \frac{\partial}{\partial t_2}\bigg|_{t_2=0}\left\{\frac{\partial}{\partial t_1}\bigg|_{t_1=0} f(p + t_2 x_2 + t_1 x_1)\right\} \\ &= \frac{\partial^2}{\partial t_1 \partial t_2}\bigg|_{(t_1, t_2)=(0,0)} f(p + t_1 x_1 + t_2 x_2). \end{aligned}$$

For more details of the differential calculus of Banach spaces, see E. Hille and R.S. Phillips [H.P] and S. Lang [L2].

(1.22) *Curves.* If $E = \mathbf{R}$ (1-dimensional Euclidean space), then $f : U \subset E \to F$ is a **curve** in $F$. Any linear mapping $T : \mathbf{R} \to F$ is given uniquely by the value $T(1)$ at $1 \in \mathbf{R}$, since $T(\lambda) = \lambda T(1)$, $\lambda \in \mathbf{R}$. So for a differentiable mapping $f : U \subset \mathbf{R} \to F$, define a **derivative coefficient** $f'(p)$ by $f'(p) = df_p(1)$. Note that $df_p(x) = xf'(1)$, $x \in \mathbf{R}$. If $f$ is differentiable everywhere on $U$, then we get a mapping $f' : U \to F$. If $f$ is $C^2$, then $(f')' : U \to F$ is defined. If $f$ is $C^k$, then $f^{(k)} := (\cdots(f')'\cdots)'$ is defined.

(1.23) If $f : U \subset \mathbf{R} \to F$ and $g : F \to G$ are differentiable, then

$$(g \circ f)'(p) = dg_{f(p)} \circ f'(p), \qquad p \in U.$$

**1.3. Integration and the mean value theorem.** Let $I = [a, b]$ be a closed interval in $\mathbf{R}$, and let $F$ be a Banach space. For an $F$-valued function on $I$, the Riemann integral of $f$ is defined as follows:

For any division $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, take any point $\xi_i \in [t_{i-1}, t_i]$ and consider the Riemann sum $\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) \in F$. Define the width of the division $\Delta$ by $\delta(\Delta) := \max_{1 \le i \le n}(t_i - t_{i-1})$. If the limit in $F$ with respect to the norm $\| \ \|$

$$\lim_{\delta(\Delta)\to 0} \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})$$

exists, then we denote the limit by $\int_a^b f(t)\,dt \in F$, called the **(Riemann) integral** of $f$. The following proposition is the same as in the case of $F = \mathbf{R}$, and its proof is left for the reader.

**Proposition (1.24).** *For any continuous function $f : I \to F$, the integral $\int_a^b f(t)dt$ is defined.*

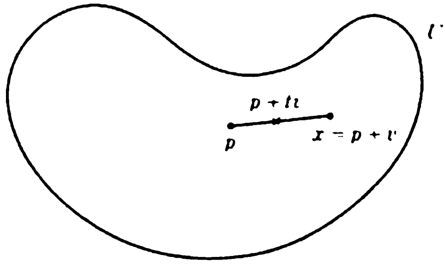



FIGURE 2.3

For a continuous function $f : I \to F$, if there exists a continuous function $g : I \to F$ such that

$$g'(t) = f(t), \qquad t \in (a, b),$$

then we call $g$ an indefinite integral. The following proposition is also the same as in the case of $F = \mathbf{R}$. We omit the proof.

PROPOSITION (1.25). *For a continuous function $f : I \to F$, define $g(x) := \int_a^x f(t)\,dt$, $x \in I$. Then $g$ is an indefinite integral of $f$.*

THE MEAN VALUE THEOREM (1.26). *Let $E$, $F$ be Banach spaces, let $p \in U \subset E$ be an open set, and let $v \in E$ be such that $p + tv \in U$, $0 \le t \le 1$ and put $x = p + v$. Let $f : U \to F$ be a $C^k$ mapping, $k \ge 1$. Then*

$$f(x) - f(p) = \int_0^1 df_{p+tv}(v)\,dt.$$

See Figure 2.3.

PROOF. We use the following Hahn-Banach theorem.

LEMMA (1.27) (Hahn-Banach). *Assume that $y \in F$ satifies $\lambda(y) = 0$ for all bounded linear mappings $\lambda : F \to \mathbf{R}$. Then $y = 0$.*

SUBLEMMA (1.27′) (Hahn-Banach). *Let $M \subset F$ be any subspace, let $\lambda : M \to \mathbf{R}$ be any bounded linear mapping, and let $x_0 \in F$ satisfy $x_0 \notin M$. Then $\lambda$ can be extended to the subspace $M + [x_0]$ without changing its norm. Here for $y \in F$, set $M + [y] = \{x + ay;\ x \in M,\ a \in \mathbf{R}\}$.*

PROOF. We shall determine $\alpha := \lambda(x_0)$ at the end and put $\lambda(x + a x_0) := \lambda(x) + a\alpha$, $x \in M$, $a \in \mathbf{R}$. We may assume $\|\lambda\| = 1$. We should determine $\alpha$ satisfying the claim with

$$|\lambda(x) + a\alpha| \le \|x + a x_0\| \quad \text{for each } x \in M \text{ and } a \ne 0.$$

Dividing both sides of the above by $a$ and rewiting the equation, we get

$$-\lambda(x_1) - \|x_1 + x_0\| \le \alpha \le -\lambda(x_2) + \|x_2 + x_0\|, \qquad x_1, x_2 \in M.$$

But by the assumption that $\|\lambda\| = 1$, it is true that the left-hand side of the above is smaller than the right-hand side because

$$\lambda(x_2) - \lambda(x_1) = \lambda(x_2 - x_1) \le \|x_2 - x_1\| \le \|x_2 + x_0\| + \|x_1 + x_0\|.$$

Therefore, we can choose $\alpha$ satisfying the desired inequality. We obtain Sublemma (1.27′).

**Proof of (1.27).** For $y \neq 0$, it suffices to show that there exists a bounded linear mapping $\lambda : F \to \mathbf{R}$ satisfying $\lambda(y) = \|y\|$ and $\|\lambda\| = 1$. First, we can define a linear mapping $\lambda : [y] \to \mathbf{R}$ with $\|\lambda\| = 1$ by $\lambda(ay) = a\|y\|$. Let $\mathfrak{X}$ be the totality of all subspaces $X$ of $F$ including $[y]$ such that $\lambda$ can be extended to $X$ with norm 1. We define a partial order $>$ on $\mathfrak{X}$ by $X > Y$ iff $X \supset Y$ for $X, Y \in \mathfrak{X}$. Then for any subset $\{X_i, i \in \Lambda\}$ of $\mathfrak{X}$ satisfying $X_i > X_j$ or $X_j > X_i$ for any $i \neq j \in \Lambda$, $\lambda$ can be extended to a subspace $\bigcup_{i\in\Lambda} X_i$ of $F$ with norm 1. Therefore, by **Zorn's lemma** there exists a maximal element $M$ in $\mathfrak{X}$ with respect to the partial order $>$. But $M$ is $F$ itself. Because if there exists an $x_0 \in F$ such that $x_0 \notin M$, then due to Sublemma (1.27′) and the definition of $M$, $\lambda$ can be extended to $M + [x_0]$ with norm 1 which contradicts the maximality of $M$. We obtain Lemma (1.27).

**Proof of Theorem (1.26).** For any bounded linear mapping $\lambda : F \to \mathbf{R}$, we have

$$\lambda\left(f(x) - f(p) - \int_0^1 df_{p+tv}(v)\,dt\right)$$
$$= (\lambda \circ f)(x) - (\lambda \circ f)(p) - \int_0^1 d(\lambda \circ f)_{p+tv}(v)\,dt.$$

Because $d(\lambda \circ f) = \lambda \circ df$ for any bounded linear mapping $\lambda : F \to \mathbf{R}$, and any $C^k$ mapping $f : U \to F$, due to (1.16) and (1.18). For a real-valued function $g(t) := (\lambda \circ f)(p + tv)$ on $[0, 1]$, $g'(t) = d(\lambda \circ f)_{p+tv}(v)$ and $g(1) - g(0) = \int_0^1 g'(t)\,dt$. Therefore, the above equation should vanish. Hence, by Lemma (1.27), we get the mean value theorem (1.26). □

By repeated application of the mean value theorem (1.26), we get the following Taylor's theorem.

**Taylor's Theorem (1.28).** *Under the same hypotheses as the mean value theorem* (1.26),

$$d^k f_{p+tv}(\underbrace{v, \dots, v}_{k \text{ times}})$$

*is continuous in $t$, and the following holds:*

$$f(x) = f(p) + \frac{1}{1!}df_p(v) + \cdots + \frac{1}{(k-1)!}d^{k-1}f_p(v, \dots, v)$$
$$+ \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} d^k f_{p+tv}(v, \dots, v)\,dt.$$

*In particular, if we put* $M := \sup_{0\le t\le 1} \|d^k f_{p+tv}\|$, *then*

$$\left\| f(x) - f(p) - \frac{1}{1!} df_p(v) - \cdots - \frac{1}{(k-1)!} d^{k-1} f_p(v, \ldots, v)\right\| \le \frac{M\|v\|^k}{k!}. \tag{1.29}$$

REMARK. Let $p \in V \subset U$, where $V$ is an open convex set, i.e., if $x, y \in V$ then $tx + (1-t)y \in V$, for each $t$ with $0 \le t \le 1$. For $m \le k$, let

$$R_m(x) := \int_0^1 \frac{(1-t)^{m-1}}{(m-1)!} d^m f_{p+t(x-p)}\, dt.$$

Then $R_m(x) \in L^m(E; F)$ and $V \ni x \mapsto R_m(x) \in L^m(E; F)$ is a $C^{m-k}$ mapping called the **remainder term**.

**1.4. The inverse function theorem.** Let $E$, $F$ be Banach spaces, and let $U \subset E$, $V \subset F$, be open subsets, respectively. A mapping $f : U \to V$ is called a **$C^k$-diffeomorphism** ($k \ge 1$) if $f : U \to V$ is one-to-one and onto, and both $f : U \to V$ and $f^{-1} : V \to U$ are $C^k$.

THE INVERSE FUNCTION THEOREM (1.30). *Let $E$, $F$ be Banach spaces, and let $U \subset E$, $V \subset F$, be open subsets, respectively. Let $f : U \to V$ be a $C^k$ mapping. Assume that at $p \in U$, $df_p : E \to F$ is a* **linear isomorphism**, i.e. , *$df_p : E \to F$ is a bounded linear mapping, and bijective, and the inverse $(df_p)^{-1} : F \to E$ is bounded. Then there exists an open set $W$ such that $p \in W \subset U$ and $f(W)$ is an open subset containing $f(p)$, and $f : W \to f(W)$ is a $C^k$-diffeomorphism.*

PROOF. Since $df_p : E \to F$ is a linear isomorphism, it is also a $C^\infty$ diffeomorphism by (1.16). Therefore, without loss of generality we may assume $E = F$, $df_p = I$, the identity mapping, and moreover, $p = 0$ and $f(p) = 0$.

Define $g : U \to E$ by $g(x) := x - f(x)$, $x \in U$. Then $dg_p = 0$. By the continuity of $g$, there exists $r > 0$ such that

$$\|x\| < 2r \quad \text{implies} \quad \|dg_x\| < \frac{1}{2}.$$

Then by the mean value theorem (1.26), we get that

$$\|x\| < 2r \quad \text{implies} \quad \|g(x)\| \le \frac{1}{2}\|x\|.$$

Therefore, if we denote by $\overline{B}_r(q) := \{x \in E;\ \|x - q\| \le r\}$ the **closed ball** centered at $p$ with radius $r$, then $g(\overline{B}_r(0)) \subset \overline{B}_{r/2}(0)$.

Here we note that any closed ball $\overline{B}_r(q)$ is a closed subset in $E$, and we can define a **distance** or **metric** $d$ on $\overline{B}_r(q)$ by

$$d(x, y) := \|x - y\|, \qquad x, y \in \overline{B}_r(q).$$

Then $(\overline{B}_r(q), d)$ is a complete metric space.

REMARK. In general, $(X, d)$ is a **metric space** if

(i) $d(x, y) = d(y, x)$, $x, y \in X$,

(ii) $d(x, y) + d(y, z) \geq d(x, z)$, $x, y, z \in X$,

(iii) $d(x, y) \geq 0$ and the equality holds if and only if $x = y$.

A sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** if $d(x_n, x_m) \to 0$ for $n, m \to \infty$. A metric space $(X, d)$ is called **complete** if every Cauchy sequence is convergent.

Now we shall show:

$$\text{For each } y \in \overline{B}_{r/2}(0), \text{ there is a unique } x \in \overline{B}_r(0) \text{ such that } f(x) = y. \tag{1.31}$$

In fact, let $g_y(z) := y + z - f(z)$. Then if $\|y\| \leq \frac{r}{2}$ and $\|z\| \leq r$, then $\|g_y(z)\| \leq r$. Therefore, $g_y$ is a mapping of the complete metric space $\overline{B}_r(0)$ into itself and satisfies

$$\begin{aligned}
\|g_y(z_1) - g_y(z_2)\| &= \|g(z_1) - g(z_2)\| \\
&= \left\| \int_0^1 dg_{z_2+t(z_1-z_2)}(z_1 - z_2)\, dt \right\| \quad \text{(by (1.26))} \\
&\leq \frac{1}{2}\|z_1 - z_2\|, \quad z_1, z_2 \in \overline{B}_r(0).
\end{aligned}$$

Therefore, by the following fixed point theorem (1.32) for a contraction mapping, $g_y$ has a unique fixed point $x$ in $\overline{B}_r(0)$: $g_y(x) = x$. Note that

$$g_y(x) = y + x - f(x) = x \quad \text{iff} \quad y = f(x), \ x \in \overline{B}_r(0)$$

which yields (1.31). Thus, we obtain the inverse mapping of $f$:

$$\varphi := f^{-1} : \overline{B}_{r/2}(0) \to \overline{B}_r(0).$$

THEOREM (1.32) (Fixed point theorem for a contraction mapping). *Let $(X, d)$ be a complete metric space, let $f : X \to X$ be a contraction mapping,* i.e., *there exists a constant $K$ with $0 < K < 1$ such that*

$$d(f(x), f(y)) \leq K\, d(x, y), \qquad x, y \in X.$$

*Then there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$. (We call such a point $x_0$, a* ***fixed point***.)

PROOF. Uniqueness. Let $x_1, x_2$ be two fixed points of $f$. Then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq K\, d(x_1, x_2);$$

thus, we get

$$0 \leq (1 - K)\, d(x_1, x_2) \leq 0. \qquad \therefore d(x_1, x_2) = 0, \qquad \therefore x_1 = x_2.$$

Existence. For $x \in X$, we shall show that $\{f^n(x) := f(\cdots(f(x))\cdots)\}_{n=1}^{\infty}$ is a Cauchy sequence in $X$. In fact,

$$d(f^{\ell+1}(x), f^{\ell}(x)) \leq K\, d(f^{\ell}(x), f^{\ell-1}(x)) \leq \cdots \leq K^{\ell}\, d(f(x), x).$$

Thus, for $m > n \geq N$,

$$\begin{aligned} d(f^m(x), f^n(x)) &\leq d(f^m(x), f^{m-1}(x)) + d(f^{m-1}(x), f^{m-2}(x)) + \cdots \\ &\quad + d(f^{n+1}(x), f^n(x)) \\ &\leq (K^{m-1} + K^{m-2} + \cdots + K^n) d(f(x), x) \\ &\leq \frac{K^n}{1-K} d(f(x), x) \leq \frac{K^N}{1-K} d(f(x), x) \to 0, \\ &\qquad \text{as } N \to \infty \end{aligned}$$

which implies $\{f^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence. But since $(X, d)$ is complete, it converges, say its limit is $x_0 := \lim_{n\to\infty} f^n(x)$. Since $f$ is continuous,

$$f(x_0) = f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = x_0.$$

We get (1.32).

Continuation of the proof of (1.30). The inverse mapping $\varphi$ of $f$ satisfies

$$\|\varphi(y_1) - \varphi(y_2)\| \leq 2\|y_1 - y_2\|, \qquad y_1, y_2 \in \overline{B}_{r/2}(0) \tag{1.33}$$

and is continuous. Because for $x_1 = \varphi(y_1)$, $x_2 = \varphi(y_2)$, $f(x_1) = y_1$, and $f(x_2) = y_2$, and since $g(x) = x - f(x)$,

$$\begin{aligned} \|x_1 - x_2\| &= \|f(x_1) - f(x_2) + g(x_1) - g(x_2)\| \\ &\leq \|f(x_1) - f(x_2)\| + \|g(x_1) - g(x_2)\| \\ &\leq \|f(x_1) - f(x_2)\| + \frac{1}{2}\|x_1 - x_2\| \end{aligned}$$

which yields that

$$\frac{1}{2}\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\|.$$

Thus, we get (1.33).

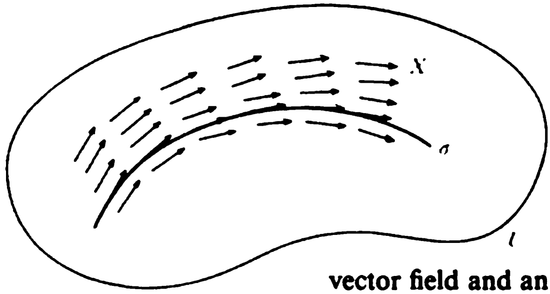
(1.34) $\varphi$ is differentiable on $B_{r/2}(0)$. Because if $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1, y_2 \in B_{r/2}(0)$, $x_1, x_2 \in \overline{B}_r(0)$, we get

$$\begin{aligned} \|\varphi(y_1) - \varphi(y_2) - df_{x_2}{}^{-1}(y_1 - y_2)\| &= \|x_1 - x_2 - df_{x_2}{}^{-1}(f(x_1) - f(x_2))\| \\ &= \|df_{x_2}{}^{-1}\| \, \|f(x_1) - f(x_2) - df_{x_2}(x_1 - x_2)\|. \end{aligned}$$

Since $\|x_1 - x_2\| \leq 2\|y_1 - y_2\|$ and $f$ is differentiable, we get

$$\begin{aligned} &\|\varphi(y_1) - \varphi(y_2) - df_{x_2}{}^{-1}(y_1 - y_2)\|/\|y_1 - y_2\| \\ &\quad \leq 2\|df_{x_2}{}^{-1}\| \, \|f(x_1) - f(x_2) - df_{x_2}(x_1 - x_2)\|/\|x_1 - x_2\| \to 0, \end{aligned}$$

if $\|y_1 - y_2\| \to 0$ since $f$ is differentiable. This implies that $\varphi$ is differentiable and $d\phi_y = df_{\phi(y)}{}^{-1}$, $y \in B_{r/2}(0)$. Since $\varphi$, $df$, $df^{-1}$ are continuous, $d\phi$ is also continuous and then $\varphi$ is $C^1$. Repeating the above arguments, we see $\varphi$ is $C^k$. □




vector field and an integral curve

FIGURE 2.4

**1.5. Ordinary differentiable equations.** Let $F$ be a Banach space, and let $U \subset F$ be an open set. We call a $C^k$ mapping $X : U \to F$, a **$C^k$-vector field** on $U$. For an open interval $I \subset \mathbf{R}$, a $C^1$-mapping $\sigma : I \to U$ is a **solution curve** (**integral curve**) of $X$ if

$$\sigma'(t) = X(\sigma(t)), \qquad t \in I.$$

If $0 \in I$, we call $\sigma(0)$ the **initial condition** of the solution $\sigma$. It is well known that

(1.35) For a continuous curve $\sigma : I \to U$ and a continuous vector field $X : U \to F$, a necessary and sufficient condition for $\sigma$ to be an integral curve of $X$ with $\sigma(0) = x_0 \in U$ is that

$$\sigma(t) = x_0 + \int_0^t X(\sigma(s))\,ds, \qquad t \in I.$$

(1.36) *Uniqueness and existence of solution.* Let $F$ be a Banach space, let $U \subset F$ be an open set, let $X$ be a $C^k$ vector field on $U$ $(k \geq 1)$, and let $p_0 \in U$. Then there exist an open subset $V$ with $p_0 \in V \subset U$ and a $C^k$ mapping $\varphi : (-\epsilon, \epsilon) \times V \to F$ such that

(i) for $p \in V$, putting $\sigma_p(t) := \varphi(t, p)$ for $t \in (-\epsilon, \epsilon)$, then $\sigma_p$ is a solution curve of $X$ at the initial condition $p$.

(ii) The solution curve $\sigma : (a, b) \to F$ of $X$ of an arbitrary initial condition $p$, satisfies

$$\sigma(t) = \sigma_p(t), \quad -\epsilon < t < \epsilon.$$

See Figure 2.4.

## §2. $C^k$-manifolds

In this section, we introduce $C^k$-manifolds, $C^k$-mappings, tangent spaces, differentials of $C^k$-mappings, vector bundles, vector fields, etc. These notions are basic and are needed for later uses. We shall give several examples of finite dimensional manifolds in §3 and of infinite dimensional manifolds in §4.

**2.1. Definition of $C^k$-manifold.** A (connected) Hausdorff topological space $M$ is called a **manifold** modelled to a Banach space $E$ if for each $p \in M$ there exist an open neighborhood $U_\alpha$ of $p$ and an into homeomorphism $\alpha : U_\alpha \to E$ such that $\alpha(U_\alpha) \subset E$. A pair $(U_\alpha, \alpha)$ is a **coordinate neighborhood** in $M$. When a collection $\{(U_\alpha, \alpha); \alpha \in A\}$ satisfies the following two conditions (i), (ii), then it is called a **$C^k$-coordinate system**, and $M$ is called a **$C^k$-manifold modelled to a Banach space E**, or simply a (**Banach**) **manifold**.

(i) $M = \bigcup_{\alpha \in A} U_\alpha$,

and

(ii) for any two coordinate neighborhoods $(U_{\alpha_1}, \alpha_1)$, $(U_{\alpha_2}, \alpha_2)$ with $U_{\alpha_1} \cap U_{\alpha_2} \neq \varnothing$, the mapping

$$\alpha_2 \circ \alpha_1^{-1} : E \supset \alpha_1(U_{\alpha_1} \cap U_{\alpha_2}) \to \alpha_2(U_{\alpha_1} \cap U_{\alpha_2}) \subset E$$

is a $C^k$-diffeomorphism.

In particular, if $E = \mathbf{R}^n$, then a $C^k$-manifold modelled to the $n$-dimensional Euclidean space $\mathbf{R}^n$ is called an **n-dimensional $C^k$-manifold**.

**2.2. $C^k$-mapping.** Let $M$, $N$ be two $C^k$-manifolds modelled to Banach spaces $E$, $F$, respectively. A mapping $\phi : M \to N$ is $C^k$ if, denoting by $\{(U_\alpha, \alpha); \alpha \in A\}$, $\{(V_\beta, \beta); \beta \in B\}$, $C^k$-coordinate neighborhood systems of $M$, $N$, respectively, for each $x \in M$, there exist $\alpha \in A$, $\beta \in B$ such that

(i) $x \in U_\alpha$, $\phi(x) \in V_\beta$,

(ii) $\phi(U_\alpha) \subset V_\beta$,

and

(iii) $\beta \circ \alpha^{-1} : E \supset \alpha(U_\alpha) \to \beta(V_\beta) \subset F$ is a $C^k$-mapping.

(2.1) If $\phi : M \to N$ is $C^k$, then $\beta' \circ \phi \circ \alpha'^{-1} : \alpha'(U_{\alpha'}) \to \beta'(V_{\beta'})$ is also $C^k$. Therefore, the above definition is independent of a choice of $\alpha \in A$, $\beta \in B$. Because

$$\beta' \circ \phi \circ \alpha'^{-1} = (\beta' \circ \beta^{-1}) \circ (\beta \circ \phi \circ \alpha^{-1}) \circ (\alpha' \circ \alpha^{-1})^{-1}$$

is also $C^k$ by the definition of $C^k$-manifolds and be (iii) above. See Figure 2.5. □

We denote by $C^k(M, N)$ the totality of $C^k$-mappings of $M$ into $N$, and if $N = \mathbf{R}$, we also denote $C^k(M) = C^k(M, \mathbf{R})$.

**2.3. Tangent spaces and differentials of mappings.**

(2.2) *Tangent spaces.* Let $p \in M$, and let $I$ be an open interval of $\mathbf{R}$ containing 0. A curve $c : I \to M$ passes through $p$ if $c(0) = p$. A curve $c$ through $p$ is $C^k$ at $p$ if for a coordinate neighborhood $(U_\alpha, \alpha)$,

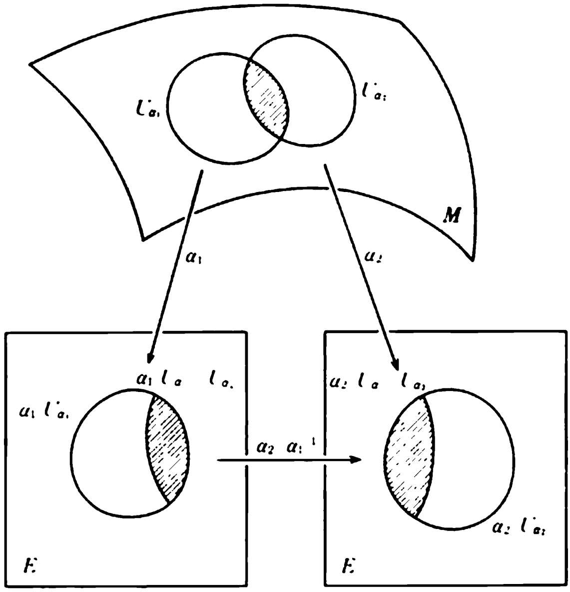



FIGURE 2.5

$\alpha \circ c : I \to \alpha(U_\alpha) \subset E$ is $C^k$. Two $C^k$-curves $c_1$, $c_2$ through $p$ are **equivalent** if for a coordinate neighborhood $(U_\alpha, \alpha)$ of $p$,

$$(\alpha \circ c_1)'(0) = (\alpha \circ c_2)'(0).$$

See Figure 2.6. We denote an equivalence class $u$ containing a $C^k$-curve $c$ through $p$, by

$$u = c'(0) = \frac{d}{dt}\bigg|_{t=0} c(t)$$

which is called a **tangent vector** at $p$. We denote by $T_pM$ the totality of all tangent vectors at $p$.

We can define naturally an addition, a scalar multiplication, and a norm on $T_pM$ which is isomorphic to the Banach space $E$, and $T_pM$ is a Banach

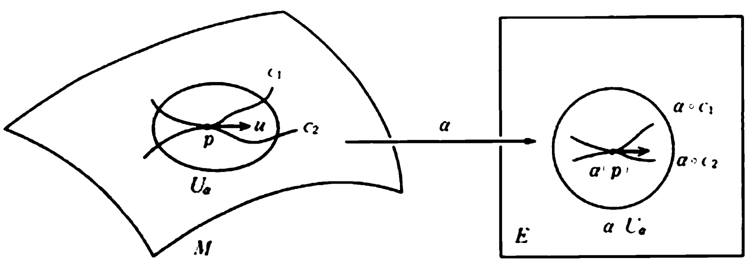



FIGURE 2.6

space itself which is called the **tangent space** of $M$ at $p$ (see also subsection 2.8).

(2.3) *Differential of a mapping.* Let $M$, $N$ be $C^k$-manifolds and let $\phi : M \to N$ be a $C^k$-mapping. For $p \in M$, define a linear mapping $d\phi_p : T_pM \to T_{\phi(p)}N$ as follows: Letting $u = c'(0) \in T_pM$ and $c$, a $C^k$-curve in $M$ through $p$, $\phi \circ c$ is a $C^k$-curve in $N$ through $\phi(p)$. Then define $d\phi_p(u) := (\phi \circ c)'(0) \in T_{\phi(p)}N$. Here note that for two $C^k$-curves $c_1$, $c_2$ contained in $u$ through $p$, if $(\alpha \circ c_1)'(0) = (\alpha \circ c_2)'(0)$, then by (1.23)

$$\begin{aligned}(\beta \circ \phi \circ c_1)'(0) &= d(\beta \circ \phi \circ \alpha^{-1})(\alpha \circ c_1)'(0) \\ &= d(\beta \circ \alpha^{-1})(\alpha \circ c_2)'(0) \\ &= (\beta \circ \phi \circ c_2)'(0)\end{aligned}$$

which implies that $d\phi_p(u)$ is determined uniquely independent of the choice of elements $c_1$, $c_2$ in $u$. This linear mapping $d\phi_p : T_pM \ni u \mapsto d\phi_p(u) \in T_{\phi(p)}N$ is called the **differential** of $\phi$ at $p$ and is denoted by $d\phi_p$ or $\phi_{*p}$.

## 2.4. Vector bundles and the induced bundles.

(2.4) *Vector bundles.* For two $C^k$-manifolds $E$, $M$, $E$ is a **$C^k$-vector bundle** over $M$ if

(i) there exists a $C^k$-mapping, say $\pi$, of $E$ onto $M$, (called the **projection**)

(ii) for each $p \in M$, $\pi^{-1}(p) =: E_p$ (called the **fiber** over $p$) has a Banach space structure, and

(iii) (local triviality) for each point $p_0 \in M$, there exist a neighborhood $U$ in $M$ of $p_0$ and a $C^k$-diffeomorphism $\Psi$ of $U \times E_{p_0}$ onto $\pi^{-1}(U)$ such that

$$\pi(\Psi(p, v)) = p, \qquad p \in U,\ v \in E_{p_0},$$

and for each $p \in U$, the mapping $E_{p_0} \ni v \mapsto \Psi(p, v) \in E_p$ gives a linear isomorphism between two Banach spaces $E_{p_0}$ and $E_p$.

(2.5) *Bundle mapping.* For another $C^k$ vector bundle $\pi' : E' \to M'$, a $C^k$-mapping $f : E \to E'$ is called to be **$C^k$-bundle mapping** if for each $p \in M$, there is a unique $p' \in M'$ such that $f(E_p) \subset E'_{p'}$ and $f : E_p \to E'_{p'}$ is a bounded linear mapping. In this case, defining $\tilde{f}(p) = p'$, a mapping $\tilde{f} : M \to M'$ is induced from $f$, which is still a $C^k$-mapping. It is called the **induced mapping** from $f$.

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\tilde{f}} & M' \end{array}$$

(2.6) *Induced bundle.* Given a $C^k$-vector bundle $\pi : E \to M$ and a $C^k$-mapping $\phi : M' \to M$ from another $C^k$-manifold $M'$, we can construct the following vector bundle $\pi' : E' \to M'$

$$E' := \{(p', v) \in M' \times E ; \phi(p') = \pi(v)\}, \qquad \pi'((p', v)) := p'.$$

That is, the fiber of this bundle over $p' \in M'$ is $E_{\phi(p')}$ the fiber of $E$ over $\phi(p')$. We denote $E'$ by $\phi^* E$ or $\phi^{-1} E$ and call it the **induced vector bundle** of $E$ by $\phi$. Its local triviality can be seen by considering that $\phi^{-1}(U) \times E_{\phi(p_0)} \cong \pi'^{-1}(\phi^{-1}(U))$ if $U \times E_{\phi(p_0)} \cong \pi^{-1}(U)$.

$$\begin{array}{ccc} \phi^{-1}E & \longrightarrow & E \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ M' & \xrightarrow{\phi} & M \end{array}$$

(2.7) *Cross section.* For a $C^k$-vector bundle $\pi : E \to M$, a $C^k$-mapping $s : M \to E$ is a **$C^k$-(cross) section** if $\pi \circ s = \mathrm{id}$; that is, $\pi(s(p)) = p$, $p \in M$. A $C^k$-section $s$ of the induced bundle $\phi^{-1}E$ of $E$ by $\phi : M' \to M$ is by definition a $C^k$-mapping $s : M' \to E$ satisfying $s(p') \in E_{\phi(p')}$, $p' \in M'$.

**2.5. Tangent bundle.**

(2.8) For a $C^{k+1}$-manifold, let $T(M) := \bigcup_{p \in M} T_p M$, and define $\pi : T(M) \to M$ by $\pi(T_p M) = p$, $p \in M$. Then $T(M)$ is a $C^k$-manifold, and $\pi : T(M) \to M$ is $C^k$-vector bundle. In fact, for $p \in M$, take a coordinate $(U_\alpha, \alpha)$, $\alpha : U_\alpha \to E$, and let $\lambda_q : T_q \to E$ be the linear isomorphism in (2.2) for each $q \in U_\alpha$. We define $\Psi_\alpha(q, v) := \lambda_q^{-1}(v)$. Then it turns out that this is a homeomorphism giving the local triviality (iii) of (2.4) $\Psi_\alpha : U_\alpha \times E \ni (q, v) \mapsto \Psi_\alpha(q, v) \in \pi^{-1}(U_\alpha) \subset T(M)$ and also giving a local coordinate of $T(M)$ (see also subsection 2.8). $T(M)$ is called the **tangent bundle** of $M$.

(2.9) Let $M$, $N$ be $C^{k+1}$-manifolds, and let $\phi : M \to N$ be a $C^{k+1}$-mapping. Then a mapping $d\phi : T(M) \to T(N)$, which is denoted also by $\phi_*$, is given by $d\phi(v) = d\phi_p(v)$, $v \in T_p M$. Then $d\phi$ is a $C^k$-bundle mapping inducing $\phi : M \to N$.

**2.6. Submanifolds and product manifolds.**

(2.10) Let $D$ be an open subset of a $C^k$ manifold $M$, and let $\{(U_\alpha, \alpha); \alpha \in A\}$ be a coordinate system of $M$. Then $\{(U_\alpha \cap D, \alpha); \alpha \in A\}$ gives a coordinate system of $D$, and $D$ becomes a $C^k$-manifold called a **$C^k$-open submanifold** of $M$.

(2.11) For a closed set $N$ of a $C^k$ manifold $M$, if $\{(U_\alpha \cap N, \alpha); \alpha \in A\}$ gives a coordinate system of $N$, i.e., there exist two closed subspaces $F$, $E'$ of a Banach space $E$ such that $E = F \oplus E'$, i.e., $E = F + E'$, and $F \cap E' = \{0\}$ and each $\alpha : U_\alpha \cap N \to \alpha(U_\alpha \cap N) \subset F$ is a heomomorphsm and satisfies the conditions (i), (ii) of subsection 2.1 for $N$, we call $N$ a **$C^k$-closed submanifold**. Then the inclusion mapping $\iota : N \subset M$ is $C^k$.

(2.12) For two $C^k$-manifolds $M$, $N$, let $\{(U_\alpha, \alpha); \alpha \in A\}$, $\{(V_\beta, \beta); \beta \in B\}$ be coordinate systems of $M$, $N$, respectively. Then the direct product $M \times N := \{(p, q); p \in M, q \in N\}$ with coordinate system given by $\{(U_\alpha \times V_\beta, \alpha \times \beta); (\alpha, \beta) \in A \times B\}$ is a $C^k$-manifold called the **product manifold** of $M$ and $N$.

**2.7. Vector fields and (differential) forms.**

(2.13) A **$C^k$-vector field** on a $C^{k+1}$-manifold $M$ is a $C^k$-section $X$, i.e., a $C^k$-mapping $X : M \to T(M)$ satisfying $\pi \circ X = \mathrm{id}$, i.e., $X(p) \in T_pM$, $p \in M$. The value $X(p)$ of $X$ at $p$ is also denoted by $X_p \in T_pM$.

(2.14) For $p \in M$, let $T_p^*M := L(T_pM, \mathbf{R})$, which is a Banach space with addition, scalar multiplication, and norm given by (1.6). $T_p^*M$ is called the **cotangent space** of $M$ at $p$. $T^*(M) := \bigcup_{p\in M} T_p^*M$, which is also a vector bundle over $M$, is called the **cotangent bundle**. A $C^k$-cross section is called **1-(differential) form.**

(2.15) For $s \geq 1$, an integer, we denote by $\bigwedge^s T_p^*M$, $p \in M$, the totality of all $s$-tuple linear mappings

$$\omega : \underbrace{T_p \times \cdots \times T_pM}_{s} \to \mathbf{R},$$

satisfying the condition

$$\omega(u_{\sigma(1)}, \dots, u_{\sigma(s)}) = \mathrm{sign}(\sigma)\, \omega(u_1, \dots, u_s), \qquad u_i \in T_pM,\ 1 \leq i \leq s,$$

for any permutation $\sigma$ of $\{1, \dots, s\}$ and $\mathrm{sign}(\sigma)$ is its sign. Then it is a closed subspace of a Banach space $L^s(T_pM; \mathbf{R})$. Moreover $\bigwedge^s T^*(M) := \bigcup_{p\in M} \bigwedge^s T_p^*M$ is a $C^k$-vector bundle over $M$. A $C^k$-cross section is called a **$C^k$-(differential) form.**

(2.16) Let $X_1, \dots, X_s$ be $s$ $C^k$-vector fields, and let $\omega$ be a $C^k$-$s$-form. Then $p \mapsto \omega_p(X_1(p), \dots, X_s(p))$ is a $C^k$-function on $M$.

(2.17) In general, considering the tensor space $T_p^{r,s}M := \bigotimes^r T_pM \otimes \bigotimes^s T_p^*M$, $p \in M$, we get the tensor bundle $T^{r,s}M$ whose $C^k$-section is called a **$C^k$-tensor field of type** $(r, s)$. An $s$-form is a tensor field of type $(0, s)$ satisfying the alternating condition in (2.15).

(2.18) Let $\phi : M \to N$ be a $C^{k+1}$-mapping. Then for a $C^k$-tensor filed $\omega$ of type $(0, s)$ on $N$, we can define the same type $C^k$-tensor field $\phi^*\omega$ on $M$ by

$$(\phi^*\omega)(u_1, \dots, u_s) := \omega(d\phi(u_1), \dots, d\phi(u_s)), \; u_1, \dots, u_s \in T_pM.$$

$\phi^*\omega$ is called the **pull back** of $\omega$ by $\phi$.

(2.19) Given a $C^k$-$r$-form $\omega$ and a $C^k$-$s$-form $\eta$ on $M$, we define a $C^k$-$(r+s)$-form $\omega \wedge \eta$, called **exterior product** of $\omega$ and $\eta$, by

$$\begin{aligned}(\omega \wedge \eta)&(X_1, \dots, X_r, X_{r+1}, \dots, X_{r+s}) \\ &:= \sum (\operatorname{sign}\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(r)})\, \eta(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)}),\end{aligned}$$

for $r+s$ $C^k$-vector fields $X_1, \dots, X_{r+s}$ on $M$. Here the sum on the right-hand side runs over all permutations $\sigma$ of $\{1, \dots, r+s\}$.

**2.8. Vector fields and coordinate neighborhoods.** In this subsection, we give an alternative definition of vector fields using coordinate systems.

First note that a Banach space $E$ is itself a $C^\infty$-manifold and that for each point $p \in E$, the tangent space $T_pE$ at $p$ can be identified with $E$ itself by considering $c'(0) = v$ where a curve $c(t) = p + tv$ through $p$ for each $v \in E$.

Now we can give an alternative definition of the tangent space $T_pM$ at $p \in M$ of a $C^{k+1}$-manifold $M$.

(2.20) *Tangent space.* Let $M$ be a $C^{k+1}$-manifold and take a point $p \in M$. For $u, v \in E$, $u$, and $v$ are called to be **equivalent** if we can choose coordinate neighborhoods $(U_\alpha, \alpha)$, $(U_\beta, \beta)$ of $p \in M$ with

$$d(\beta \circ \alpha^{-1})u = v,$$

where $d(\beta \circ \alpha^{-1})$ is a differential in the sense of subsection 1.2 at $\alpha(p)$ of a $C^{k+1}$ diffeomorphism $\beta \circ \alpha^{-1} : E \supset \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta) \subset E$. Define a tangent vector at $p$ by its equivalence class, and define the tangent space $T_pM$ at $p$ to be the totality of all equivalence classes. Namely, if we take a coordinate neighborhood $(U_\alpha, \alpha)$, where $\alpha : U_\alpha \to \alpha(U_\alpha) \subset E$ is a homeomorphism, then $E$ can be regarded as the tangent space of $M$ at $p$. If we take another coordinate neighborhood $(U_\beta, \beta)$ of $p$, we should identify according to the above relation.

This definition of the tangent space is equivalent to the one in (2.2), and the local triviality of $T(M)$: $\pi^{-1}(U_\alpha) \cong U_\alpha \times E$ is automatic, i.e., for a fixed coordinate neighborhood $(U_\alpha, \alpha)$, all tangent vectors at each point $q \in U_\alpha$ are determined by both the point $q$ itself and a vector in $E$. This gives the local triviality. See Figure 2.7, next page.

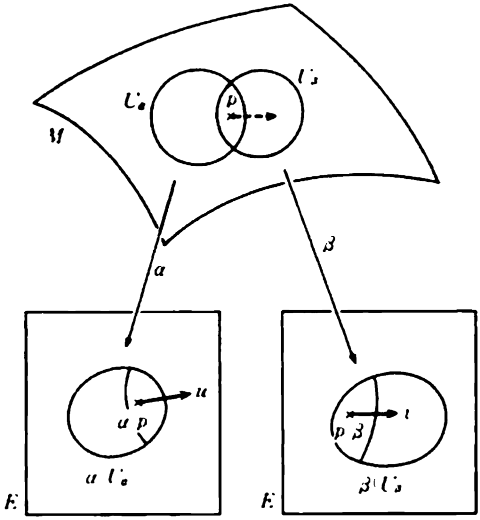


FIGURE 2.7

(2.21) A $C^k$-vector field $X$ is defined by taking a coordinate neighborhood $(U_\alpha, \alpha)$ and writing

$$X(q) = X_\alpha(\alpha(q)), \qquad q \in U_\alpha,$$

where $X_\alpha : \alpha(U_\alpha) \to E$ is a $C^k$-mapping satisfying the condition

$$d(\beta \circ \alpha^{-1})_{\alpha(q)} X_\alpha(\alpha(q)) = X_\beta(\beta(q)), \qquad q \in U_\alpha \cap U_\beta,$$

for two coordinate neighborhoods $(U_\alpha, \alpha)$, $(U_\beta, \beta)$, with $U_\alpha \cap U_\beta \neq \emptyset$.

We denote also by $X_p$, the value $X(p) \in T_pM$ of $X$ at $p$. For a $C^{k+1}$-function $f : M \to \mathbf{R}$, if we denote by $df : TM \to T\mathbf{R}$ the differential of $f$, then for all $C^k$-vector fields $X$ we can define a $C^k$-function $Xf$ by

$$Xf(p) = X_pf := df_p \cdot X(p), \qquad p \in M.$$

Here the right-hand side is the $X(p) \in T_pM$-direction derivative of $f$ at $p \in M$ and we identify $T_{f(p)}\mathbf{R} \cong \mathbf{R}$. That $Xf$ is $C^k$ is proved as follows: that $f$ is $C^{k+1}$ is by the definition in subsection 2.2, $f_\alpha := f \circ \alpha^{-1} : \alpha(U_\alpha) \to \mathbf{R}$ is $C^{k+1}$ and

$$X_pf = df_p \cdot X(p) = df_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p))$$

which is $C^k$ since $X_\alpha : \alpha(U_\alpha) \to E$ is $C^k$.

(2.22) *Commutator of vector fields.* For two $C^k$-vector fields $X$, $Y$ on $M$, we define a $C^{k-1}$-vector field $[X, Y]$, called a **commutator** or **bracket**, as follows. For $p \in M$, $[X, Y]_p \in T_pM$ is a tangent vector which is represented by

$$dY_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p)) - dX_{\alpha_{\alpha(p)}} \cdot Y_\alpha(\alpha(p)),$$

for a coordinate neighborhood $(U_\alpha, \alpha)$ of $p \in M$. Here $dY_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p))$ is the $X_\alpha(\alpha(p))$-direction differentiation of a $C^k$-mapping $Y_\alpha : E \supset \alpha(U_\alpha) \to E$ at a point $\alpha(p)$, and $dX_{\alpha_{\alpha(p)}} \cdot Y_\alpha(\alpha(p))$ is defined in the same way. If we take another coordinate neighborhood $(U_\beta, \beta)$, then we can calculate directly

$$\begin{aligned} d(\beta \circ \alpha^{-1})(dY_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p)) - dX_{\alpha_{\alpha(p)}} \cdot Y_\alpha(\alpha(p))) \\ = dY_{\beta_{\beta(p)}} \cdot X_\beta(\beta(p)) - dX_{\beta_{\beta(p)}} \cdot Y_\beta(\beta(p)). \end{aligned}$$

Moreover, for each $f \in C^k(M)$,

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf). \tag{2.23}$$

In fact, the left-hand side is by definition,

$$df_\alpha \cdot (dY_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p))) - df_\alpha \cdot (dX_{\alpha_{\alpha(p)}} \cdot Y_\alpha(\alpha(p))).$$

The $X_p(Yf)$ of the right-hand side is by definition, $d(df_\alpha \cdot Y_\alpha) \cdot X_\alpha(\alpha(p))$, which satisfies

$$\begin{aligned} d(df_\alpha \cdot Y_\alpha) \cdot X_\alpha(\alpha(p)) = d^2 f_\alpha(Y_\alpha(\alpha(p)), X_\alpha(\alpha(p))) \\ + df_\alpha \cdot (dY_{\alpha_{\alpha(p)}} \cdot X_\alpha(\alpha(p))). \end{aligned}$$

By (1.20), the first term satisfies

$$d^2 f_\alpha(Y_\alpha(\alpha(p)), X_\alpha(\alpha(p))) = d^2 f_\alpha(X_\alpha(\alpha(p)), Y_\alpha(\alpha(p))).$$

The similar equations hold for $Y_p(Xf)$, and we get (2.23). □

REMARK. One can use (2.23) for an alternative definition of $[X, Y]$. But in this case, one must check that $f \mapsto X_p(Yf) - Y_p(Xf)$ gives a tangent vector at $p$. We shall find this troublesome to do in the case where $\dim M = \infty$.

(2.24) *Exterior differentiation of a form.* For a $C^k$-$r$-form $\omega$, we define a $C^{k-1}$-$(r+1)$-form $d\omega$ by

$$\begin{aligned} (d\omega)(X_1, \dots, X_{r+1}) := \sum_{i=1}^{r+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{r+1})) \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{r+1}), \end{aligned}$$

for $r+1$ $C^k$-vector fields $X_1, \dots, X_{r+1}$. Here we write $\widehat{X}_i$ to indicate that $X_i$ is deleted. Then one can check that

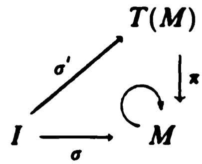
$$d(d\omega) = 0. \tag{2.25}$$

**2.9. Integral curve of a vector field.** Let us recall that a $C^1$-curve $\sigma$ in a $C^{k+1}$-manifold $M$ is by definition a $C^1$-mapping of an open interval $I = (a, b)$ (an 1-dimensional manifold) into $M$. Then we get a continuous mapping $\sigma' : I \to T(M)$ by

$$\sigma'(t) = d\sigma_t(1), \qquad t \in I.$$

For the mapping $\pi : T(M) \to M$ in (2.8), $\pi(\sigma'(t)) = \sigma(t)$, $t \in I$.

(2.26) A $C^1$-curve $\sigma : I \to M$ is an **integral curve (solution curve)** of a $C^k$-vector field $X$ if $\sigma' = X \circ \sigma$, i.e., $d\sigma_t(1) = \sigma'(t) = X(\sigma(t)) \in T_{\sigma(t)}M$, $t \in I$. When $0 \in I$, $\sigma(0)$ is called the **initial condition** of $\sigma$.




THEOREM (2.27). *Let $k \geq 1$. For a $C^k$-vector field $X$ on a $C^{k+1}$-manifold $M$ and $p \in M$, there exists an integral curve $\sigma_p$ of $X$ whose initial condition is $p$ and satisfies the following condition: Any integral curve $\sigma$ of $X$ with the initial condition $p$ is a restriction of $\sigma_p$. (This $\sigma_p$ is called the* **maximal integral curve** *of $X$ with the initial condition $p$. We denote the existence interval $I$ of $\sigma_p$ by $(t^-(p), t^+(p))$.)*

OUTLINE OF THE PROOF. Using a coordinate neighborhood $U_\alpha$, $\alpha : U_\alpha \to E$, transform a $C^1$-curve $\sigma$ through $p$ to a curve $\alpha \circ \sigma : I \to E$ in $E$. Then the condition $\sigma' = X \circ \sigma$ for $\sigma$ to be an integral curve of $X$ is transformed to the corresponding condition for $\alpha \circ \sigma$ which is given by

$$(\alpha \circ \sigma)'(t) = d\alpha_{\sigma(t)}(X(\alpha^{-1}(\alpha \circ \sigma(t)))).$$

Therefore, if we define a vector field $X_\alpha$ on $\alpha(U_\alpha) \subset E$ by

$$X_\alpha(z) := d\alpha_{\alpha^{-1}(z)}X(\alpha^{-1}(z)), \qquad z \in \alpha(U_\alpha) \subset E,$$

then $\alpha \circ \sigma$ satisfies

$$(\alpha \circ \sigma)' = X_\alpha \circ (\alpha \circ \sigma).$$

That is, $\alpha \circ \sigma$ is an integral curve of $X_\alpha$. Together with the uniqueness and existence (1.36) of a solution of a differential equation, and using the relation $d(\beta \circ \alpha^{-1})_{\alpha(q)}X_\alpha(\alpha(q)) = X_\beta(\beta(q))$, $q \in U_\alpha \cap U_\beta$ in (2.21) for $X_\alpha$ and $X_\beta$, we obtain the existence of such a $\sigma_p$. See Figure 2.8.

By Theorem (2.27), we immediately obtain

COROLLARY (2.28). *We put $q := \sigma_p(s)$ for a fixed $s \in (t^-(p), t^+(p))$. Let $\sigma_q$ be a maximal integral curve of $X$ with the initial condition $q$. Then*

$$\sigma_q(t) = \sigma_p(t + s)$$

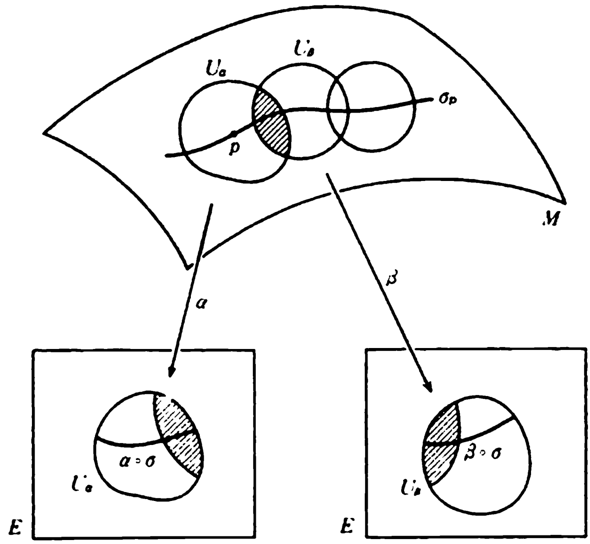



FIGURE 2.8

*and* $t^+(q) = t^+(p) - s$, $t^-(q) = t^-(p) - s$.

THEOREM (2.29). (i) *If* $t^+(p) < \infty$, *then* $\sigma_p(t)$ *has no accumulation point in* $M$ *as* $t$ *tends to* $t^+(p)$.

(ii) *If* $-\infty < t^-(p)$, *then* $\sigma_p(t)$ *has no accumulation point in* $M$ *as* $t$ *tends to* $t^-(p)$.

COROLLARY (2.30). *If* $M$ *is compact, then* $t^+(p) = \infty$, *and* $t^-(p) = -\infty$ *for any* $p \in M$.

PROOF OF COROLLARY (2.30). Assume that there exists a $p \in M$ such that $t^+(p) < \infty$. Take a sequence $\{t_n\}$ such that $t_n \to t^+(p) < \infty$ (as $n \to \infty$). Then since $t^+(p) < \infty$, we may assume that $\{\sigma_p(t_n)\}_{n=1}^{\infty}$ is bounded. Since $M$ is compact, $\{\sigma_p(t_n)\}_{n=1}^{\infty}$ has a convergent subsequence, say $\{\sigma_p(t_{n_k})\}_{k=1}^{\infty}$. Then $t_{n_k} \to t^+(p)$ and $\sigma(t_{n_k}) \to y_0 \in M$ as $k \to \infty$, which contradicts to (i) of Theorem (2.29). See Figure 2.9. We get (ii) in a similar manner. □

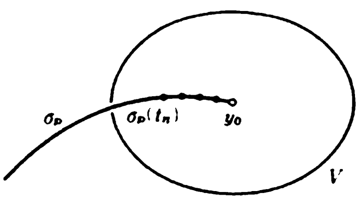



FIGURE 2.9

**Proof of Theorem** (2.29). Assume that there exists $\{t_n\}$ such that $t_n \to t^+(p) < \infty$ and $\sigma_p(t_n) \to y_0 \in M$. Taking a sufficiently small neighborhood $V$ of $y_0$ and a sufficiently small open interval $I = (-\delta, \delta)$ such that for all $z \in V$, we find that there exists an integral curve $\sigma_z : I \to M$ with the initial condition $z$ by (1.36). For a large $n$, we can take $t_n$ such that

$$t^+(p) - \delta < t_n < t^+(p) \quad \text{and} \quad \sigma_p(t_n) \in V.$$

Considering a maximal integral curve $\sigma_z : I \to M$ with the initial condition $z = \sigma_p(t_n)$, we see that a curve $t \mapsto \sigma_p(t + t_n)$ is also an integral curve with initial condition $z$. By uniqueness, it follows that $\sigma_z(t) = \sigma_p(t + t_n)$ for some interval containing $0$. So we define

$$\sigma(t) := \begin{cases} \sigma_p(t), & t^-(p) < t < t^+(p), \\ \sigma_z(t - t_n), & t^+(p) \le t < t_n + \delta. \end{cases}$$

Then $\sigma$ is an integral curve of $X$ with initial condition $p$, and its defining domain is larger than interval $(t^-(p), t^+(p))$ which is a contradition. □

## 2.10. Riemannian metric.

(2.31) *Hilbert manifold.* Let $M$ be a $C^{k+1}$-manifold, and let $(H, (\,,\,))$ be a separable Hilbert space, i.e., it has a basis consisting of a countable number of elements. If $M$ is a $C^{k+1}$-manifold modelled to $(H, (\,,\,))$, then $M$ is called a **Hilbert manifold**. Since the Euclidean space $(\mathbf{R}^n, (\,,\,))$ is an $n$-dimensional Hilbert space, any $n$-dimensional manifold is a Hilbert manifold.

(2.32) *Riemannian metric.* If a $C^k$-tensor $g$ of type (0,2) on $M$ satisfies

(i) $g_p(u, v) = g_p(v, u)$, $u, v \in T_pM$, $p \in M$, and

(ii) $g_p(u, u) \ge 0$, and the equality holds if and only if $u = 0$, then

we call $g$ a **$C^k$-Riemannian metric** on $M$, and $(M, g)$ is called a **$C^k$-Riemannian manifold**. We write $g_p(u, u)^{1/2} = \|u\|$ for brevity.

If we take a coordinate neighborhood $(U_\alpha, \alpha)$, $\alpha : U_\alpha \to H$, for each $x \in U_\alpha$, then using $d\alpha : T_xM \to H$ we can express

$$g_x(u, v) = (G^\alpha(x)\, d\alpha(u), d\alpha(v)), \qquad u, v \in T_xM.$$

Here $G^\alpha(x) : H \to H$ is a positive definite operator of $(H, (\,,\,))$; that is,

$$(G^\alpha(x)w, w) \ge 0, \quad w \in H, \quad \text{and the equality holds if and only if } w = 0.$$

That $g$ is $C^k$ means that the mapping $U_\alpha \ni x \mapsto G_\alpha(x) \in L(H, H)$ is $C^k$.

The inner product $g_p$ on $T_pM$, $p \in M$, induces the ones on the cotangent space $T_p^*M$, $p \in M$. The tensor space $T_p^{r,s}M$, $p \in M$, of type $(r, s)$ is denoted by the same letter $g_p$ or simply by $(\,,\,)$. We denote the corresponding norms by $\|\ \|$.

(2.33) *Length of a curve.* For a $C^1$-curve $\sigma : [a, b] \to M$, i.e., for a sufficiently small $\epsilon > 0$, $\sigma$ is extended to a $C^1$-mapping of $(a - \epsilon, b + \epsilon)$ into $M$, the function $[a, b] \ni t \mapsto \|\sigma'(t)\|$ is continuous, and then the **length** of $\sigma$

$$L(\sigma) := \int_a^b \|\sigma'(t)\| \, dt$$

is well defined.

(2.34) *Riemannian distance.* For each pair of points $x, y \in M$, we define $\rho(x, y)$ by

$$\rho(x, y) := \inf\{L(\sigma);\ \sigma \text{ is a } C^1\text{-curve connecting } x \text{ and } y\},$$

(here we assume $M$ is connected, i.e., $x$ and $y$ can always be connected by a finite number of $C^1$-curves.) Then $\rho$ satisfies the three axioms of distance:

(i) $\rho(x, y) = \rho(y, x)$, $x, y \in M$,

(ii) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$, $x, y, z \in M$,

(iii) $\rho(x, y) \geq 0$, and the equality holds if and only if $x = y$.

Moreover, the topology on $M$ induced from this distance coincides with the original topology of $M$. If $(M, \rho)$ is complete as a metric space; that is, any Cauchy sequence $\{x_n\}_{n=1}^\infty$, i.e., $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$, is convergent, then we call $(M, g)$ **complete**. If $M$ is compact, then $(M, g)$ is always complete.

## §3. Finite-dimensional $C^\infty$-manifolds

For the remainder of this book, a $C^\infty$-manifold always means a finite dimensional $C^\infty$-manifold. We shall always indicate when we are considering an infinite dimensional manifold .

**3.1. Local coordinates.** Let $M$ be an $n$-dimensional $C^\infty$-manifold, and let $(x_1, \dots, x_n)$ be the standard coordinates of $\mathbf{R}^n$. For any local coordinate neighborhood $(U_\alpha, \alpha)$ of $M$, where $\alpha : U_\alpha \to \mathbf{R}^n$, define a **local coordinate** $(x_1^\alpha, \dots, x_n^\alpha)$ by

$$x_i^\alpha := x_i \circ \alpha : U_\alpha \to \mathbf{R}, \qquad i = 1, \dots, n.$$

Then each point of $U_\alpha$ can be uniquely expressed by the coordinate $(x_1^\alpha, \dots, x_n^\alpha)$. We often simply write $U$, $(x_1, \dots, x_n)$ for the coordinate neighborhood and its local coordinate, omitting $\alpha$.

(3.1) *Coordinate expression of vector field.* Let $U_\alpha$, $(x_1^\alpha, \dots, x_n^\alpha)$, be the local coordinate of $M$, and then for $p \in U_\alpha$ we denote by $(x_1^\alpha(p), \dots, x_n^\alpha(p)) = (a_1, \dots, a_n)$. Then we consider a $C^1$-curve $c_i$ through $p$ defined by

$$c_i(t) := (a_1, \dots, a_{i-1}, a_i + t, a_{i+1}, \dots, a_n),$$

and we denote by $(\frac{\partial}{\partial x_i^\alpha})_p$, its tangent vector $c_i'(0)$ at $p$. Then

$$\left\{\left(\frac{\partial}{\partial x_1^\alpha}\right)_p, \dots, \left(\frac{\partial}{\partial x_n^\alpha}\right)_p\right\}$$

is a basis of the tangent space $T_pM$ of $M$ at $p$. A $C^\infty$-vector field $X$ on $M$ is written on $U_\alpha$ as

$$X_p = \sum_{i=1}^n \xi_i^\alpha(p)\left(\frac{\partial}{\partial x_i^\alpha}\right)_p, \qquad p \in U_\alpha,$$

where $\xi_i^\alpha$, $i = 1, \dots, n$, are in $C^\infty(U_\alpha)$. Moreover, if we take another coordinate neighborhood $U_\beta$, $(x_1^\beta, \dots, x_n^\beta)$ and denote also on $U_\beta$,

$$X = \sum_{i=1}^n \xi_i^\beta\left(\frac{\partial}{\partial x_i^\beta}\right),$$

then both $(\xi_1^\alpha, \dots, \xi_n^\alpha)$, $(\xi_1^\beta, \dots, \xi_n^\beta)$ satisfy on $U_\alpha \cap U_\beta$,

$$\xi_i^\beta = \sum_{j=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha}\xi_j^\alpha.$$

Compare this to (2.21). We denote by $\mathfrak{X}(M)$ the totality of all $C^\infty$-vector fields on $M$.

For $X$, $Y \in \mathfrak{X}(M)$, denoting $X$ and $Y$ on $U_\alpha$ by

$$X = \sum_{i=1}^n \xi_i^\alpha \frac{\partial}{\partial x_i^\alpha}, \qquad Y = \sum_{i=1}^n \eta_i^\alpha \frac{\partial}{\partial x_i^\alpha},$$

we see the commutator $[X, Y]$ of $X$, $Y$ is given by

$$\begin{aligned}[X, Y] &= \sum_{i=1}^n \left\{d\eta_i^\alpha(X) - d\xi_i^\alpha(Y)\right\}\frac{\partial}{\partial x_i^\alpha} \\ &= \sum_{i=1}^n \left\{\sum_{j=1}^n \left(\xi_j^\alpha \frac{\partial \eta_i^\alpha}{\partial x_j^\alpha} - \eta_j^\alpha \frac{\partial \xi_i^\alpha}{\partial x_j^\alpha}\right)\right\}\frac{\partial}{\partial x_i^\alpha},\end{aligned}$$

because $\left[\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha}\right] = 0$ and by definition (2.22).

(3.2) *Local expression of forms.* For a $C^\infty$-function $f$ on a coordinate neighborhood $U_\alpha$, we defined in (2.21) a $C^\infty$-1-form $df$ on $U_\alpha$ by $(df)_p : T_pM \to T_{f(p)}\mathbf{R} = \mathbf{R}$, $p \in U_\alpha$. For a coordinate $(x_1^\alpha, \dots, x_n^\alpha)$ on $U_\alpha$, $\{(dx_i^\alpha)_p\}_{i=1}^n$ is a basis for the cotangent space $T_p^*M$ because

$$(dx_i^\alpha)_p\left(\left(\frac{\partial}{\partial x_j^\alpha}\right)_p\right) = \frac{\partial x_i^\alpha}{\partial x_j^\alpha} = \delta_{ij}, \qquad 1 \le i, j \le n.$$

Then $df$, $f \in C^\infty(M)$ can be written on $U_\alpha$ as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^\alpha} dx_i^\alpha.$$

We denote by $A^r(M)$ the totality of all $C^\infty$-$r$-forms on $M$. Each $\omega \in A^r(M)$ can be written on a coordinate neighborhood $U_\alpha$ using $(x_1^\alpha, \dots, x_n^\alpha)$,

$$\begin{aligned}\omega &= \frac{1}{r!} \sum_{i_1, \dots, i_r = 1}^{n} \omega_{i_1 \dots i_r} dx_{i_1}^\alpha \wedge \cdots \wedge dx_{i_r}^\alpha \\ &= \sum_{1 \le i_1 < \cdots < i_r \le n} \omega_{i_1 \cdots i_r} dx_{i_1}^\alpha \wedge \cdots \wedge dx_{i_r}^\alpha,\end{aligned}$$

where $\omega_{i_1 \cdots i_r} \in C^\infty(U_\alpha)$. Since, for $p \in U_\alpha$,

$$(dx_{i_1}^\alpha)_p \wedge \cdots \wedge (dx_{i_r}^\alpha)_p, \qquad 1 \le i_1 < \cdots < i_r \le n$$

is a basis of $\bigwedge^r T_p^* M$. Moreover, the exterior differentiation $d\omega$ of $\omega \in A^r(M)$ is calculated by

$$d\omega = \frac{1}{r!} \sum_{i_1, \cdots, i_r = 1}^{n} d\omega_{i_1 \cdots i_r} \wedge dx_{i_1}^\alpha \wedge \cdots \wedge dx_{i_r}^\alpha.$$

(3.3) *Local expression of a Riemannian metric.* Assume that $g$ is a $C^\infty$-Riemannian metric on $M$. Write $g_{ij}^\alpha := g(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha})$ using coordinates $(x_1^\alpha, \cdots, x_n^\alpha)$ on $U_\alpha$. The $n \times n$-matrix $(g_{ij}^\alpha)$ is a positive definite symmetric matrix and

$$g = \sum_{i,j=1}^{n} g_{ij}^\alpha dx_i^\alpha dx_j^\alpha.$$

Here $dx_i^\alpha dx_j^\alpha = dx_i^\alpha \otimes dx_j^\alpha$ is a quadratic form $T_pM \times T_pM \to \mathbf{R}$ defined by

$$dx_i^\alpha \otimes dx_j^\alpha(u, v) = dx_i^\alpha(u) dx_j^\alpha(v), \qquad u, v \in T_pM.$$

### 3.2. Levi-Civita connection.

(3.4) *Connection.* A **connection (covariant differentiation)** $\nabla$ on a $C^\infty$-manifold $M$ is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M),$$

satisfying the following conditions:

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
(2) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
(3) $\nabla_{fX} Y = f \nabla_X Y$,
(4) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

for $f \in C^\infty(M)$, $X$, $Y$, $Z \in \mathfrak{X}(M)$. Due to (3), it turns out that the value $(\nabla_X Y)_p \in T_pM$ of $\nabla_X Y$ at $p \in M$ depends only on $u = X_p \in T_pM$ and $Y$, it can be written as $\nabla_u Y$.

THEOREM (3.5). *Let $(M, g)$ be an n-dimensional $C^\infty$-Riemannian manifold. Then a connection $\nabla$ (called the* **Levi-Civita connection**) *can be given*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]), \end{aligned} \tag{3.6}$$

*for $X, Y, Z \in \mathfrak{X}(M)$. Moreover, the Levi-Civita connection $\nabla$ satisfies*

(i) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

*and*

(ii) $\nabla_X Y - \nabla_Y X - [X, Y] = 0$.

*Conversely, any connection $\nabla$ satisfying* (i), (ii) *coincides with the Levi-Civita connection.*

Using local coordinates $U$, $(x_1, \dots, x_n)$, we put

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \qquad \Gamma_{ij}^k \in C^\infty(U),$$

where by (3.6) and $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, the functions $\Gamma_{ij}^k$ are given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^n g^{k\ell} \left( \frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{i\ell}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell} \right). \tag{3.7}$$

Here let $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, and let $(g^{k\ell})$ denote the inverse matrix of $(g_{ij})$. $\Gamma_{ij}^k$ is called **Christoffel's symbol** of the Levi-Civita connection.

**3.3. Parallel displacement.** For a $C^1$-curve $\sigma : [a, b] \to M$, $X$ is called a $C^1$-vector field along $\sigma$ if

(1) $X(t) \in T_{\sigma(t)}M$, for all $t \in [a, b]$,

and

(2) if we express $X(t) = \sum_{i=1}^n \xi_i(t) (\frac{\partial}{\partial x_i})_{\sigma(t)}$ using local coordinates $U$, $(x_1, \dots, x_n)$, then each $\xi_i(t)$ is $C^1$.

Such a vector field $X$ along $\sigma$ is called **parallel** along $\sigma$ if

$$\nabla_{\sigma'(t)} X = 0, \qquad t \in (a, b). \tag{3.8}$$

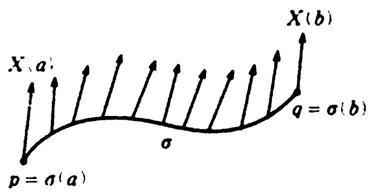



FIGURE 2.10

See Figure 2.10. A parallel vector field $X$ along $\sigma$ is uniquely determined by the initial value $X(a)$ at $\sigma(a)$ in the following way: In terms of local coordinates $(x_1, \dots, x_n)$, we write the curve $\sigma$ by $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$, then we have $\sigma'(t) = \sum_{i=1}^n \sigma'(t)(\frac{\partial}{\partial x_i})_{\sigma(t)}$. By properties (1)–(4) of $\nabla$, (3.8) is equivalent to

$$\frac{d\xi_i(t)}{dt} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(\sigma(t))\frac{d\sigma_j(t)}{dt}\xi_k(t) = 0, \qquad i = 1, \dots, n. \tag{3.9}$$

Therefore, if a curve $\sigma$ and the initial condition $(\xi_1(a), \cdots, \xi_n(a))$ at $p = \sigma(a)$ are given, then such a $\xi_i(t)$ is uniquely determined due to the existence and uniqueness of solutions of the ordinary differential equation. In particular, the value $(\xi_1(b), \dots, \xi_n(b))$ at $q = \sigma(b)$, and hence, $X(b)$ are uniquely determined.

Therefore, in particular, we obtain the correspondence

$$T_{\sigma(a)}M \ni X(a) \mapsto X(b) \in T_{\sigma(b)}M,$$

which is denoted by $P_\sigma$. Then this mapping

$$P_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M$$

is a linear isomorphism and satisfies

$$g_{\sigma(b)}(P_\sigma(u), P_\sigma(v)) = g_{\sigma(a)}(u, v), \qquad u, v \in T_{\sigma(a)}M.$$

Because, in (i) of Theorem (3.5), substituting for $Y$, $Z$, two parallel vector fields along $\sigma$ satisfying $Y(a) = u$, $Z(a) = v$ and for $X = \sigma'(t)$, we get

$$\frac{d}{dt}g_{\sigma(t)}(Y(t), Z(t)) = g(\nabla_{\sigma'(t)}Y, Z) + g(Y, \nabla_{\sigma'(t)}Z) = 0$$

which implies that $g_{\sigma(t)}(Y(t), Z(t))$ is constant in $t$. □

This mapping $P_\sigma$ is called the **parallel displacement (transport)** along $\sigma$.

Using the parallel displacement, the covariant derivative $\nabla$ can be defined in the following way. Indeed, for $u \in T_pM$, $X \in \mathfrak{X}(M)$, $\nabla_u X \in T_pM$, $p \in M$ satisfies (cf. [K.N])

$$\nabla_u X = \left.\frac{d}{dt}\right|_{t=0} a(t), \tag{3.10}$$

where $\sigma$ is a $C^1$-curve in $M$ satisfying $\sigma(0) = p$, $\sigma'(0) = u$, and for all $t$,

$$\sigma_t(s) := \sigma(s), \qquad 0 \le s \le t.$$

$P_{\sigma_t} : T_pM \to T_{\sigma(t)}M$ is the parallel displacement along $\sigma_t$. Then the curve $a(t)$ in (3.10) is the $C^1$-curve in $T_pM$ given by

$$a(t) := P_{\sigma_t}^{-1}X(\sigma(t)) \in T_pM.$$

The differentiation of the right-hand side of (3.10) expresses the tangent vector of the curve $a(t)$ in $T_p$ at $t = 0$. The equation (3.10) can be regarded as the definition of the covariant derivative.

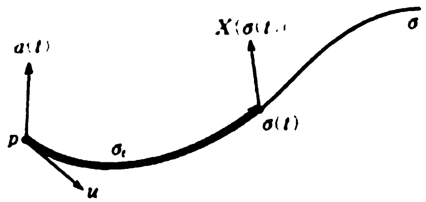



FIGURE 2.11

**3.4. Geodesics.** A $C^1$-curve in $M$, $\sigma : I \to M$ is a **geodesic** if

$$\nabla_{\sigma'}\sigma' = 0, \tag{3.11}$$

on each point of the open interval $I$. (See Figure 2.11.)

In terms of local coordinates $(x_1, \ldots, x_n)$ of $M$, denoting $\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))$ and $\sigma'(t) = \sum_{i=1}^n \sigma'(t)(\frac{\partial}{\partial x_i})_{\sigma(t)}$, we see that (3.11) reduces to the equation

$$\frac{d^2\sigma_i}{dt^2} + \sum_{j,k=1}^n \Gamma^i_{jk}\frac{d\sigma_j}{dt}\frac{d\sigma_k}{dt} = 0, \qquad i = 1, \ldots, n, \tag{3.12}$$

in the same way as for (3.9). Putting $\xi_i := \frac{d\sigma_i}{dt}$, (3.12) is reduced to the following ordinary differential equations in the unknown functions $(\xi_1, \ldots, \xi_n)$,

$$\frac{d\xi_i}{dt} = -\sum_{j,k=1}^n \Gamma^i_{jk}\xi_j\xi_k, \qquad i = 1, \ldots, n. \tag{3.13}$$

Therefore, for given initial data $(\sigma_1(0), \ldots, \sigma_n(0))$ and $(\frac{d\sigma_1}{dt}(0), \ldots, \frac{d\sigma_n}{dt}(0))$, there exists a unique solution for all $t$ near 0. That is, for given $p \in M$ and $u \in T_pM$, there exists a unique geodesic $\sigma(t)$ for small $|t|$ satisfying $\sigma(0) = p$ and $\sigma'(0) = u$, denoted by $\sigma(t) = \exp_p(tu)$. By putting $\sigma(1) = \exp_p u \in M$ for $u \in T_pM$, the **exponential mapping**

$$\exp_p : T_pM \to M$$

is well defined on a neighborhood of 0 in $T_pM$. Concerning the question of when for all $u \in T_pM$, the geodeisc $\exp_p(tu)$ can be extended to $-\infty < t < \infty$, the following theorem is well known.

THEOREM (3.15) (Hopf-Rinow). *The following are equivalent:*

(i) $(M, g)$ *is complete (cf.* (2.34)).

(ii) *For any* $p \in M$, *the exponential mapping* $\exp_p : T_pM \to M$ *is well defined everywhere on* $T_pM$. *Therefore, in this case, any two points* $p, q \in M$ *can be joined by a geodesic of length* $\rho(p, q)$ (*if* $M$ *is connected*).

By theorem (3.15), if $M$ is compact, then for any $p \in M$, $\exp_p : T_pM \to M$ is well defined everywhere on $T_pM$.

**3.5. Curvature tensor.** For an $n$-dimensional $C^\infty$-Riemannian manifold $(M, g)$, $R(X, Y)Z \in \mathfrak{X}(M)$, $X, Y, Z \in \mathfrak{X}(M)$ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \tag{3.16}$$

where $\nabla$ is the Levi-Civita connection on $(M, g)$. $R$ is a tensor field of type $(1,3)$ and satisfies

$$R(fX, gY)(hZ) = fgh\, R(X, Y)Z, \qquad f, g, h \in C^\infty(M). \tag{3.17}$$

$R$ is called the **curvature tensor field** of $(M, g)$. By (3.17), $R(X, Y)Z$ depends only on $u = X(p)$, $v = Y(p)$, $w = Z(p) \in T_pM$ at $p \in M$. We can write $R(u, v)w = (R(X, Y)Z)_p \in T_pM$.

In terms of local coordinates $(x_1, \dots, x_n)$ of $M$, writing

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_{\ell=1}^{n} R^{\ell}_{kij} \frac{\partial}{\partial x_\ell}, \qquad 1 \le i, j, k \le n,$$

we get

$$R^{\ell}_{kij} = \frac{\partial}{\partial x_i}\Gamma^{\ell}_{kj} - \frac{\partial}{\partial x_j}\Gamma^{\ell}_{ki} + \sum_{a=1}^{n}\left\{\Gamma^{a}_{kj}\Gamma^{\ell}_{ai} - \Gamma^{a}_{ki}\Gamma^{\ell}_{aj}\right\}.$$

(3.18) *Curvature.* For any two linearly independent vectors $\{u, v\}$ of the tangent space $T_pM$ of $M$ at $p$, we define

$$K(u, v) := \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}.$$

$K(u, v)$ is called the **sectional curvature** along $\{u, v\}$. $(M, g)$ is **positive curvature** (**negative curvature**) if for all $p \in M$ and two linearly independent vectors $\{u, v\}$ of $T_pM$, $K(u, v) \ge 0$ $(K(u, v) \le 0)$.

(3.19) *Ricci operator.* The tensor field $\rho : T_pM \to T_pM$, $p \in M$ of type $(1,1)$, called the **Ricci operator**, transform, is well defined by

$$\rho(u) := \sum_{i=1}^{n} R(u, e_i)e_i, \qquad u \in T_pM, \ p \in M,$$

where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for $(T_pM, g_p)$. And the symmetric tensor field of type (0,2), denoted by the same letter $\rho$ and defined by

$$\rho(u, v) := g(\rho(u), v) = g(u, \rho(v)) = \sum_{i=1}^{n} g(R(u, e_i)e_i, v),$$

is called the **Ricci tensor**.

The $C^\infty$-function $S$ on $M$ defined by $S := \sum_{i=1}^{n} \rho(e_i, e_i)$, is called the **scalar curvature** of $(M, g)$ and is independent of the choice of orthonormal basis $\{e_i\}$.

**3.6. Integration.** In this subsection, we define an **integral** $\int_M f v_g$ of any continuous function $f$ on an $n$-dimensional $C^\infty$-Riemannian manifold $(M, g)$. We always assume $M$ is compact, but integration can be defined, in general, for a paracompact manifold $M$, i.e., for any open covering there exists its locally finite refinement. In any case, we use a **partition of unity** $\{\phi_\alpha; \alpha \in A\}$ for a coordinate system $\{(U_\alpha, \alpha); \alpha \in A\}$. That is, $\phi_\alpha \in C^\infty(M)$ satisfying

(i) $0 \le \phi_\alpha(x) \le 1$, $x \in M$, $\alpha \in A$,

(ii) the support $\operatorname{supp}(\phi_\alpha) := \overline{\{x \in M; \phi_\alpha(x) \neq 0\}}$ is contained in $U_\alpha$ for all $\alpha \in A$, where the overline denotes the topological closure, and

(iii) for any point $x \in M$, $\sum_{\alpha\in A} \phi_\alpha(x) = 1$.

Now for a coordinate neighborhood $(U_\alpha, \alpha)$ and for a continuous function $f$ on $U_\alpha$, we define

$$\int_{U_\alpha} f v_g := \int_{\alpha(U_\alpha)} f_\alpha \sqrt{g}\, dx_1^\alpha \cdots dx_n^\alpha, \qquad f_\alpha := f \circ \alpha^{-1},$$

where $g := \det\left(g\left(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha}\right)\right)$, and the right-hand side is the usual Lebesgue integration of a continuous function $f_\alpha \sqrt{g}$ on an open set $\alpha(U_\alpha)$ of $\mathbf{R}^n$.

In general, for a continuous function $f$ on $M$ (we assume $\operatorname{supp}(f) := \overline{\{x \in M; f(x) \neq 0\}}$ is compact in $M$ if $M$ is noncompact), we define integrate of $f$ by

$$\int_M f v_g := \int_M \sum_{\alpha\in A} \phi_\alpha f v_g = \sum_{\alpha\in A} \int_M \phi_\alpha f v_g,$$

using the partition of unity $\{\phi_\alpha; \alpha \in A\}$. In fact, it is well defined. Because, if $f$ is expressed in two ways as: $f = \sum_i f_i = \sum_j g_j$, where $f_i$ satisfies for each $i$, there is an $\alpha$ such that $\operatorname{supp}(f_i) \subset U_\alpha$, and $g_j$ satisfies the same condition, then for all $\alpha \in A$, since

$$\sum_i f_i \phi_\alpha = \sum_j g_j \phi_\alpha \qquad \text{and} \qquad \operatorname{supp}(g_j \phi_\alpha),\ \operatorname{supp}(f_i \phi_\alpha) \subset U_\alpha,$$

we get

$$\sum_i \int_M f_i \phi_\alpha v_g = \sum_j \int_M g_j \phi_\alpha v_g.$$

On the other hand, since

$$f_i = \sum_{\alpha\in A} f_i \phi_\alpha, \qquad g_j = \sum_{\alpha\in A} g_j \phi_\alpha,$$

by the assumptions on $f_i$, $g_j$, we can define $\int_M f_i v_g$ and $\int_M g_j v_g$ and get

$$\int_M f_i v_g = \sum_{\alpha\in A} \int_M f_i \phi_\alpha v_g \quad \text{and} \quad \int_M g_j v_g = \sum_{\alpha\in A} \int_M g_j \phi_\alpha v_g.$$

Therefore, we obtain

$$\begin{aligned}\sum_i \int_M f_i v_g &= \sum_i \sum_{\alpha\in A} \int_M f_i \phi_\alpha v_g = \sum_{\alpha\in A}\sum_i \int_M f_i\phi_\alpha v_g \\ &= \sum_{\alpha\in A}\sum_j \int_M g_j\phi_\alpha v_g = \sum_j\sum_{\alpha\in A}\int_M g_j\phi_\alpha v_g = \sum_j \int_M g_j v_g\end{aligned}$$

which implies that $\int_M f v_g$ is well defined independently of the expression for $f$.

If $f \equiv 1$, then $\mathrm{Vol}(M) := \int_M v_g$ is called the **volume** of $(M, g)$. If $M$ is not compact, then the volume is not necessarily finite.

For $f \in C^0(M)$, i.e., a continuous function $f$ on $M$, we define the norm $\| \ \|_p$, $0 < p < \infty$, by

$$\|f\|_p := \left(\int_M |f|^p v_g\right)^{1/p} \tag{3.21}$$

and denote by $L_p(M)$ the Banach space obtained by the completion of $C^0(M)$ with respect to the norm $\| \ \|_p$. We define the norm $\| \ \|_\infty$ on $C^0(M)$ by

$$\|f\|_\infty := \sup\{|f(x)|\,;\, x \in M\}, \qquad f \in C^0(M). \tag{3.22}$$

**3.7. Divergence of a vector field and the Laplacian.** For a $C^\infty$-vector field $X \in \mathfrak{X}(M)$, we define $\mathrm{div}(X) \in C^\infty(M)$ by

$$\mathrm{div}(X)(p) := \sum_{i=1}^n g(e_i, \nabla_{e_i} X)(p), \qquad p \in M, \tag{3.23}$$

where $\{e_i\}_{i=1}^n$ are $n$ $C^\infty$-vector fields on a local coordinate neighborhood $U$ satisfying the condition that $\{e_i(x)\}_{i=1}^n$ is an orthonormal basis for $(T_xM, g_x)$ for each point $x \in U$ (i.e., a **local orthonormal frame field**). The existence of such a frame field can be shown by using the Gramm-Schmit process of orthonormalization on $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ on a local coordinate neighborhood $(U, (x_1, \dots, x_n))$. Due to the form of its normalization, the $\{e_i\}_{i=1}^n$ are $C^\infty$ on $U$. The right-hand side of (3.23) does not depend on the choice of $\{e_i\}_{i=1}^n$ and defines an element in $C^\infty(M)$, called the **divergence** of $X$.

If we write $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ in terms of local coordinates $(x_1, \cdots, x_n)$, then it can be shown (cf. exercise 2.3) that

$$\mathrm{div}(X) = \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial}{\partial x_i}(\sqrt{g}\, X_i), \tag{3.24}$$

where $g = \det\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)$.

(3.25) *Gradient vectors.* For $f \in C^\infty(M)$, the **gradient vector field** $X = \mathrm{grad}\, f = \nabla f \in \mathfrak{X}(M)$ is the one satisfying

$$g(Y, X) = df(Y) = Yf, \qquad Y \in \mathfrak{X}(M).$$

Then it can be shown (cf. exercise 2.4) that

$$\operatorname{grad} f = \nabla f = \sum_{i=1}^{n} e_i(f)\, e_i = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}. \tag{3.26}$$

We also denote by the same letter $g$ the induced inner product on $T_p^*M$, $p \in M$. Then we get (cf. exercise 2.4)

$$g(\operatorname{grad} f_1, \operatorname{grad} f_2) = g(d f_1, d f_2), \qquad f_1, f_2 \in C^\infty(M).$$

Next we define the important differential operator.

(3.28) *Laplacian.* We define (cf. exercise 2.5) a second order elliptic diffential operator acting on $C^\infty(M)$, called the **Laplacian**, by

$$\begin{aligned}\Delta f &= -\operatorname{div} \operatorname{grad} f \\ &= -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j}\right) \\ &= -\sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}\right) \\ &= -\sum_{i=1}^{n} \left\{ e_i(e_i f) - (\nabla_{e_i} e_i) f \right\}, \qquad f \in C^\infty(M).\end{aligned}$$

The Laplacian depends on a Riemannian metric $g$, so we write $\Delta_g$ if we wish to emphasize a Riemannian metric $g$. The following holds if $M$ is compact.

**Proposition** (3.29). *The following hold for* $f, f_1, f_2 \in C^\infty(M)$, $X \in \mathfrak{X}(M)$,

(i) $\int_M f \operatorname{div}(X)\, v_g = -\int_M g(\operatorname{grad} f, X)\, v_g$,
(ii) $\int_M (\Delta f_1) f_2\, v_g = \int_M g(\operatorname{grad} f_1, \operatorname{grad} f_2)\, v_g = \int_M f_1 (\Delta f_2)\, v_g$,
(iii) $\int_M \operatorname{div}(X)\, v_g = 0$ (**Green's formula**).

**Proof.** For (i), since $\nabla_{e_i}(fX) = (e_i f) X + f \nabla_{e_i} X$, we get $\operatorname{div}(fX) = g(\operatorname{grad} f, X) + f \operatorname{div}(X)$. Integrating this, we get (i) by (iii). For (ii), substituting $f = f_1$, $X = \operatorname{grad} f_2$ in (i), we get the second equality of (ii), and by $g(\operatorname{grad} f_1, \operatorname{grad} f_2) = g(\operatorname{grad} f_2, \operatorname{grad} f_1)$, we get the first equality of (ii).

For (iii), we use the partition of unity (cf. 3.6): $1 = \sum_{\alpha \in A} \phi_\alpha$. It can be seen that $X = \sum_{i=1}^n X_i^\alpha \frac{\partial}{\partial x_i^\alpha}$ on $U_\alpha$. Then

$$\int_M \operatorname{div}(X)\, v_g = \int_M \operatorname{div}\left(\left(\sum_{\alpha \in A} \phi_\alpha\right) X\right) v_g = \sum_{\alpha \in A} \int_M \operatorname{div}(\phi_\alpha X)\, v_g.$$

Here since $\operatorname{supp}(\phi_\alpha X) = \overline{\{x \in M;\, \phi_\alpha(x) X_x \neq 0\}}$ is included in $U_\alpha$, we get

$$\begin{aligned}\int_M \operatorname{div}(\phi_\alpha X)\, v_g &= \int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, v_g \\ &= \int_{U_\alpha} \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial}{\partial x_i^\alpha}(\sqrt{g}\phi_\alpha X_i^\alpha)\sqrt{g}\, dx_1^\alpha \cdots dx_n^\alpha \\ &= \sum_{i=1}^n \int_{U_\alpha} \frac{\partial}{\partial x_i^\alpha}(\sqrt{g}\phi_\alpha X_i^\alpha)\, dx_1^\alpha \cdots dx_n^\alpha,\end{aligned}$$

where for each $i = 1, \ldots, n$, $\alpha \in A$, we get

$$\int_{U_\alpha} \frac{\partial}{\partial x_i^\alpha}(\sqrt{g}\phi_\alpha X_i^\alpha)\, dx_1^\alpha \cdots dx_n^\alpha = 0.$$

Because the integral $\int \frac{\partial}{\partial x_i^\alpha}(\sqrt{g}\phi_\alpha X_i^\alpha)\, dx_i^\alpha$ equals to the value of $\sqrt{g}\phi_\alpha X_i^\alpha$ at the boundary of $U_\alpha$, which vanishes since $\operatorname{supp}(\phi_\alpha X) \subset U_\alpha$, we obtain (iii). □

**3.8. Laplacian for forms.** We denote by $A^r(M) = \Gamma(\bigwedge^r T^*M)$ the totality of all $C^\infty$-sections of the $C^\infty$-vector bundle $\bigwedge^r T^*M$, which admits a canonical inner product induced from $g$ on each fiber $\bigwedge^r T_p^*M$, $p \in M$, denoted by $(\,,\,)$. We assume in this subsection that $M$ is compact. Then $A^r(M)$ admits the inner product $(\ ,\ )$ defined by

$$(\omega, \eta) := \int_M (\omega, \eta)\, v_g, \qquad \omega, \eta \in A^r(M). \tag{3.30}$$

The **codifferentiation** $\delta : A^{r+1}(M) \to A^r(M)$ of the exterior differentiation $d : A^r(M) \to A^{r+1}(M)$ is by definition

$$(d\omega, \eta) = (\omega, \delta\eta), \qquad \omega \in A^r(M),\ \eta \in A^{r+1}(M). \tag{3.31}$$

In fact, the codifferentiation $\delta\eta$, $\eta \in A^{r+1}(M)$ is given (cf. Exercise 2.6) by

$$\delta\eta(X_1, \ldots, X_r) = -\sum_{j=1}^n (\nabla_{e_j}\eta)(e_j, X_1, \ldots, X_r), \qquad X_1, \ldots, X_r \in \mathfrak{X}(M), \tag{3.32}$$

where $\{e_j\}_{j=1}^n$ is a locally defined orthonormal frame field as in subsection 3.7. We explain the $\nabla_X\eta$, $X \in \mathfrak{X}(M)$ in the right-hand side: In general, the mapping

$$A^r(M) \ni \omega \mapsto \nabla_X\omega \in A^r(M), \qquad X \in \mathfrak{X}(M)$$

is called the **covariant derivative** for forms and is defined by

$$\begin{aligned}(\nabla_X\omega)(X_1, \ldots, X_r) := {}& X(\omega(X_1, \ldots, X_r)) \\ & - \sum_{i=1}^r \omega(X_1, \ldots, \nabla_X X_i, \ldots, X_r).\end{aligned}$$

Then it is known (cf. exercise 2.7) by definition of $d\omega$ and (ii) of Theorem 3.5, that for $\omega \in A^r(M)$,

$$(d\omega)(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i+1}(\nabla_{X_i}\omega)(X_1, \dots, \widehat{X}_i, \dots, X_{r+1}).$$

We define the **Laplacian** acting on $r$-forms by

$$\Delta_r := d\delta + \delta d \colon A^r(M) \to A^r(M).$$

By definition, we have that

$$(\Delta_r\omega, \eta) = (\omega, \Delta_r\eta), \qquad \omega, \eta \in A^r(M). \tag{3.37}$$

In particular, for $r = 0$ $f \in A^0(M) = C^\infty(M)$,

$$\delta_0 f = \delta d f = -\sum_{i=1}^{n}\nabla_{e_i}(df)(e_i) = -\sum_{i=1}^{n}\{e_i(e_i f) - \nabla_{e_i}e_i f\} = \Delta_g f.$$

For more details about the Laplacian of $r$-forms, see [Mt].

## §4. Examples of manifolds

### 4.1. Finite-dimensional $C^\infty$-Riemannian manifolds.

(4.1) *Euclidean space* $\mathbf{R}^n$. The Euclidean space $\mathbf{R}^n$ is an $n$-dimensional $C^\infty$-manifold, and if $(x_1, \dots, x_n)$ are the standard coordinates on $\mathbf{R}^n$, then

$$g_0 := \sum_{i=1}^{n} dx_i \otimes dx_i$$

is a Riemannian metric on $\mathbf{R}^n$ whose sectional curvature is zero.

(4.2) *Flat torus* $\mathbf{R}^n/\Lambda$. Let $\{v_1, \dots, v_n\}$ be a basis for $\mathbf{R}^n$, and let

$$\Lambda = \left\{\sum_{i=1}^{n} m_i v_i ;\ m_i \in \mathbf{Z}\,(i = 1, \dots, n)\right\}.$$

$\Lambda$ is called a **lattice** of $\mathbf{R}^n$. See Figure 2.12.

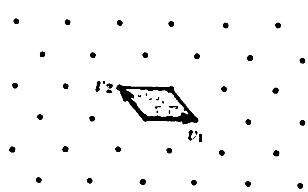



FIGURE 2.12

We say two elements $x$, $y \in \mathbf{R}^n$ are equivalent if $x - y \in \Lambda$, and we denote by $\mathbf{R}^n/\Lambda$ the totality of all equivalence classes $\pi(x)$, $x \in \mathbf{R}^n$. Then $\mathbf{R}^n/\Lambda$ is an $n$-dimensional compact $C^\infty$-manifold. We may take as its local coordinates, $\mathbf{R}^n/\Lambda \ni \pi(y) = \pi(\sum_{i=1}^n y_i v_i) \mapsto (y_1, \dots, y_n)$. We may define a Riemannian metric $g_\Lambda$ on $\mathbf{R}^n/\Lambda$ by

$$g_\Lambda = \sum_{i,j=1}^n g_{ij}\, dy_i \otimes dy_j,$$

where $g_{ij} = (v_i, v_j)$. Then $\pi^* g_\Lambda = g_0$, where $\pi$ is the projection of $\mathbf{R}^n \to \mathbf{R}^n/\Lambda$ (cf. exercise 2.8). The resulting compact Riemannian manifold $(\mathbf{R}^n/\Lambda, g_\Lambda)$, called a **flat torus**, has zero sectional curvature.

(4.3) *Unit sphere* $S^n$. The **unit sphere**

$$S^n := \left\{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1};\ \sum_{i=1}^{n+1} x_i^2 = 1\right\} \subset \mathbf{R}^{n+1}$$

is an $n$-dimensional compact $C^\infty$-manifold and a closed submanifold of $\mathbf{R}^{n+1}$. We may define a Riemannian metric $g_{S^n}$ on $S^n$ by the pull back $g_{S^n} := \iota^* g_0$ by the inclusion $\iota : S^n \subset \mathbf{R}^{n+1}$. The sectional curvature of $(S^n, g_{S^n})$ is one (cf. (2.17) of subsection 2.1 in Chapter 4).

(4.4) *Unit ball* $B^n$. The **unit open ball**

$$B^n := \left\{(x_1, \dots, x_n) \in \mathbf{R}^n;\ \sum_{i=1}^{n} x_i^2 < 1\right\}$$

is a noncompact $C^\infty$-manifold which is $C^\infty$-diffeomorphic to $\mathbf{R}^n$. The Riemannian metric $g_{B^n}$ on $B^n$, defined by

$$g_{B^n} := \frac{4}{1 - \sum_{i=1}^n x_i^2} \sum_{i=1}^n dx_i \otimes dx_i,$$

has constant sectional curvature $-1$. There are an infinite number of $n$-dimensional compact $C^\infty$-Riemannian manifolds $(M, g)$ such that $B^n$ is the universal covering of $M$ and $g_{B^n} = \pi^* g$, where $\pi : B^n \to M$ is the covering mapping. The sectional curvatures of all such manifolds $M$ are always $-1$.

Here in general, an $n$-dimensional $C^\infty$-manifold $\widetilde{M}$ is the **universal covering** of a $C^\infty$-manifold $M$ of the same dimension if $\widetilde{M}$ is simply connected, i.e., the fundamental group $\pi_1(\widetilde{M}) = \{0\}$ (see (4.45), below), and there exits the following $C^\infty$-mapping of $\widetilde{M}$ onto $M$, $\pi : \widetilde{M} \to M$ (called the **covering mapping**) : For each point $p \in M$ there exists a neighborhood $V$ of $p$ such that

$$\pi^{-1}(V) = \bigcup_i V_i, \qquad V_i \cap V_j = \emptyset \quad (i \neq j),$$

and for each $i$, $\pi : V_i \to V$ is an onto $C^\infty$-diffeomorphism, respectively.

Moreover, if the Riemannian metrics $\tilde{g}$, $g$ on $\widetilde{M}$, $M$, respectively, satisfy $\tilde{g} = \pi^* g$, then we say $(\widetilde{M}, \tilde{g})$ is a **Riemannian covering** of $(M, g)$. The Euclidean space $(\mathbf{R}^n, g_0)$ with $\pi : (\mathbf{R}^n, g_0) \to (\mathbf{R}^n/\Gamma, g_\Gamma)$ is such an example.

**4.2. Lie groups and homogeneous spaces.** Here we introduce Lie groups and homogeneous spaces which give interesting examples of manifolds. For details, see [Mt].

(4.5) DEFINITION. If a group $G$ is itself an $n$-dimensional $C^\infty$-manifold and the group actions

$$G \times G \ni (x, y) \mapsto xy \in G, \qquad G \ni x \mapsto x^{-1} \in G$$

are both $C^\infty$ mappings, then $G$ is called a **Lie group**. We denote by $e$ the identity element of $G$. The $C^\infty$-mappings $L_x$, $R_x : G \to G$ defined by

$$L_x y := xy, \qquad R_x y := yx, \qquad x, y \in G,$$

are called the **left translation**, **right translation** by the element $x$, respectively. $L_x$ and $R_x$ are $C^\infty$-diffeomorphisms of $G$ onto itself.

A $C^\infty$-vector field $X \in \mathfrak{X}(G)$ on $G$ is called **left invariant** if

$$L_{x*} X_y = X_{xy}, \qquad x, y \in G,$$

where we denote by $X_x \in T_x G$ the value in $T_x G$ of $X$ at $x \in G$. $X$ is called **right invariant** if $R_{x*} X_y = X_{yx}$, $x, y \in G$. We denote by $\mathfrak{g}$ the totality of all left invariant $C^\infty$-vector fields on $G$. $\mathfrak{g}$ is a subspace of $\mathfrak{X}(G)$. The mapping defined by

$$\mathfrak{g} \ni X \mapsto X_e \in T_e G \tag{4.6}$$

is bijective and $\dim(\mathfrak{g}) = \dim(T_e G) = \dim G$. Surjectiveness of (4.6) follows. Let $v \in T_e G$. If we define

$$X_x := L_{x*} v \in T_x G, \qquad x \in G,$$

then $X \in \mathfrak{g}$ and $X_e = v$.

For $X$, $Y \in \mathfrak{g}$, since their commutator $[X, Y]$ satisfies

$$L_{x*}[X, Y] = [L_{x*} X, L_{x*} Y], \qquad x \in G,$$

$[X, Y]$ belongs to $\mathfrak{g}$. This $\mathfrak{g}$ is called the **Lie algebra** of $G$.

Here we explain the exponential mapping of the Lie algebra $\mathfrak{g}$ into the Lie group $G$. For $X \in \mathfrak{g}$, let $\sigma(t)$ be an integral curve of $X$ with the initial condition $e$. That is,

$$\sigma(0) = e, \qquad \sigma'(t) = X_{\sigma(t)}.$$

This equation has a solution for all $-\infty < t < \infty$. Because, if for $|t| < \epsilon$ there exists a solution, then we can extend it by $\tilde{\sigma}(t + t_0) := \sigma(t_0)\sigma(t)$, for $0 < t_0 < \epsilon$. Then $\tilde{\sigma}(t_0) = \sigma(t_0)$, and since $X$ is left invariant, it follows that

$$\tilde{\sigma}'(t + t_0) = L_{\sigma(t_0)*}\sigma'(t) = L_{\sigma(t_0)*} X_{\sigma(t)} = X_{\sigma(t_0)\sigma(t)} = X_{\tilde{\sigma}(t+t_0)}$$

which implies that $\tilde{\sigma}$ is an integral curve of $X$ through $\sigma(t_0)$ at $t = t_0$. By the uniqueness (2.27) of the solution of an integral curve of $X$, we get $\tilde{\sigma}(t+t_0) = \sigma(t+t_0)$, for $0 < t+t_0 < \epsilon$. Therefore $\sigma$ can be extended beyond the open interval $(-\epsilon, \epsilon)$.

We denote this integral curve $\sigma(t)$, $-\infty < t < \infty$, by $\exp(tX)$. By definition, it follows that

$$\exp(tX)\exp(sX) = \exp((t+s)X), \qquad -\infty < t, s < \infty.$$

The mapping $\exp : \mathfrak{g} \ni X \mapsto \exp(X) \in G$ is called the **exponential mapping** of a Lie group. (Don't confuse it with (3.14) !)

A subgroup $K$ of a Lie group $G$ is called a **Lie subgroup** if $K$ is a submanifold of a manifold $G$. In particular, if $K$ is a closed submanifold of $G$, it is called a **closed Lie subgroup**. Then if we put

$$\mathfrak{k} := \{X \in \mathfrak{g};\, \exp(tX) \in K, \text{ for each } t \in \mathbf{R}\}, \tag{4.7}$$

then $\mathfrak{k}$ satisfies $[X, Y] \in \mathfrak{k}$ for all $X, Y \in \mathfrak{k}$, and it is the Lie algebra of $K$ if we regard $K$ is a Lie group. We call $\mathfrak{k}$ the **Lie subalgebra** corresponding to $K$.

For two Lie groups $G_1$, $G_2$, a $C^\infty$-mapping $\varphi : G_1 \to G_2$ is called a **homomorphism** if

$$\varphi(xy) = \varphi(x)\varphi(y), \qquad x, y \in G_1.$$

Then the differentiation $\varphi_{*e} : T_eG_1 \to T_eG_2$ of $\varphi$ induces, via the identification (4.6), a linear mapping $d\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ which satisfies

$$\begin{cases} d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)], & X, Y \in \mathfrak{g}_1, \\ \varphi(\exp X) = \exp d\varphi(X), & X \in \mathfrak{g}_1. \end{cases} \tag{4.8}$$

If a homomorphism $\varphi : G_1 \to G_2$ is bijective and the inverse $\varphi^{-1}$ is $C^\infty$, then we say $\varphi$ is an **isomorphism**. If $G = G_1 = G_2$, then $\varphi$ is called an **automorphism** and $d\varphi : \mathfrak{g} \to \mathfrak{g}$ is called an **automorphism** of the Lie algebra $\mathfrak{g}$.

For $x \in G$, an automorphism $A_x : G \to G$ defined by

$$A_x y = L_x R_x^{-1} y = R_x^{-1} L_x y = xyx^{-1}, \quad y \in G, \tag{4.9}$$

is called an **inner automorphism**. Since $A_x e = e$, the differentiation $A_{x*} : T_eG \to T_eG$ of $A_x$ at $e$ induces an automorphism of $\mathfrak{g}$. We denote it by $\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$. We call $G \ni x \mapsto \mathrm{Ad}(x)$ the **adjoint representation** of $G$. This satisfies by (4.8),

$$\exp \mathrm{Ad}(x) X = x \exp(X) x^{-1}, \qquad x \in G,\ X \in \mathfrak{g}. \tag{4.10}$$

If $\dim(G) = n$ and $\{X_1, \dots, X_n\}$ is a basis for $\mathfrak{g}$, then for $x \in G$,

$$x \exp\left(\sum_{i=1}^n x_i X_i\right) \mapsto (x_1, \dots, x_n)$$

gives local coordinates in a neighborhood of $x$ in $G$.

For a Lie group $G$ and its closed Lie subgroup $K$, let us consider the **quotient space** defined by

$$G/K := \{xK\,;\; x \in G\}.$$

The mapping $\pi : G \ni x \mapsto xK \in G/K$ is called the projection of $G$ into $G/K$. We introduce a Hausdorff topology on $G/K$ by saying that a subset $U$ of $G/K$ is an open set if $\pi^{-1}(U)$ is open in $G$. Moreover, letting $\dim(G) = n$, $\dim(K) = s$, and taking as a basis for $\mathfrak{g}$, $\{X_1, \dots, X_s, X_{s+1}, \dots, X_n\}$ in such a way that $\{X_1, \dots, X_s\}$ is a basis of $\mathfrak{k}$, then we may give an $(n-s)$-dimensional $C^\infty$-manifold structure in such a way that

$$G/K \ni x \exp\left(\sum_{i=s+1}^{n} x_i X_i\right) K \mapsto (x_{s+1}, \dots, x_n)$$

are local coordinates on a neighborhood of $xK$. Then the mappings $G \times G/K \ni (x, yK) \mapsto xyK \in G/K$ and $\pi : G \to G/K$ are $C^\infty$. Moreover, for $x \in G$, a mapping $\tau_x : G/K \to G/K$ defined by

$$\tau_x(yK) := xyK, \qquad y \in G$$

is a $C^\infty$ diffeomorphism of $G/K$ onto itself, called the **translation** by $x$.

(4.11) EXAMPLE 1. Let us denote by $M(n, \mathbf{R})$ the totality of all $n \times n$ real matrices, and let

$$\mathrm{GL}(n, \mathbf{R}) := \{x \in M(n, \mathbf{R})\,;\, \det x \neq 0\}.$$

$\mathrm{GL}(n, \mathbf{R})$ is an open submanifold of $M(n, \mathbf{R}) = \mathbf{R}^{n^2}$. It can be shown that $\mathrm{GL}(n, \mathbf{R})$ is a Lie group, i.e., the mappings defined by

$$\mathrm{GL}(n, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R}) \ni (x, y) \mapsto xy \in \mathrm{GL}(n, \mathbf{R}),$$
$$\mathrm{GL}(n, \mathbf{R}) \ni x \mapsto x^{-1} \in \mathrm{GL}(n\mathbf{R})$$

are both $C^\infty$. The mappings $\mathrm{GL}(n, \mathbf{R}) \ni x = (x_{ij}) \mapsto x_{ij}$, $1 \le i, j \le n$, give the coordinates of $\mathrm{GL}(n, \mathbf{R})$. We denote by $\mathfrak{gl}(n, \mathbf{R})$, $M(n, \mathbf{R})$ endowed with the Lie bracket given by

$$[X, Y] := XY - YX, \qquad X, Y \in M(n, \mathbf{R}).$$

For each $X = (X_{ij}) \in \mathfrak{gl}(n, \mathbf{R})$, we can define uniquely a left invariant vector field $\tilde{X}$ on $\mathrm{GL}(n, \mathbf{R})$ by

$$\tilde{X}_a := \sum_{i,j,k=1}^{n} a_{ik} X_{kj} \left(\frac{\partial}{\partial x_{ij}}\right)_a, \qquad a = (a_{ij}) \in \mathrm{GL}(n, \mathbf{R}). \tag{4.12}$$

The linear mapping $X \mapsto \tilde{X}$ is bijective and satisfies

$$[\tilde{X}, \tilde{Y}] = [X, Y]^{\sim}, \qquad X, Y \in \mathfrak{gl}(n, \mathbf{R}),$$

which implies that $\mathfrak{gl}(n, \mathbf{R})$ can be regarded as the Lie algebra of $\mathrm{GL}(n, \mathbf{R})$. Then the exponential mapping $\exp : \mathfrak{gl}(n, \mathbf{R}) \to GL(n, \mathbf{R})$ is given by

$$\exp(\tilde{X}) = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}, \qquad X \in \mathfrak{gl}(n, \mathbf{R}),$$

see exercise 2.9.

(4.13) EXAMPLE 2. (i) The groups

$$\mathrm{O}(n) := \{x \in M(n, \mathbf{R});\ {}^t x\, x = x\, {}^t x = I\},$$
$$\mathrm{SO}(n) := \{x \in \mathrm{O}(n);\ \det x = 1\}$$

are compact closed Lie subgroups, called the **orthogonal**, the **spectial orthogonal groups**, respectively. Here we denote by ${}^t x$, $\det x$ the transposed matrix, and the determinant of a matrix $x$, respectively. $I$ is the unit matrix.

(ii) We denote by $M(n, \mathbf{C})$ the totality of all $n \times n$ complex matrices. Then

$$\mathrm{U}(n) := \{z \in M(n, \mathbf{C});\ {}^t\bar{z}\, z = z\, {}^t\bar{z} = I\},$$
$$\mathrm{SU}(n) := \{z \in U(n);\ \det z = 1\}$$

are both compact Lie groups, called the **unitary** and the **special unitary groups**, respectively. Here $\bar{z}$ implies the complex conjugate of $z \in M(n, \mathbf{C})$.

(iii) The Lie algebra of both $\mathrm{O}(n)$, $\mathrm{SO}(n)$ is

$$\mathfrak{so}(n) := \{X \in \mathfrak{gl}(n, \mathbf{R});\ {}^t X + X = O\},$$

and the Lie algebras of $\mathrm{U}(n)$, $\mathrm{SU}(n)$ are

$$\mathfrak{u}(n) := \{Z \in M(n, \mathbf{C});\ {}^t\bar{Z} + Z = O\},$$
$$\mathfrak{su}(n) := \{Z \in \mathfrak{u}(n);\ \mathrm{tr}\, Z = 0\},$$

where the Lie bracket is given by $[Z, W] = ZW - WZ$, for $Z, W \in M(n, \mathbf{C})$, and $\mathrm{tr}\, Z := \sum_{i=1}^n Z_{ii}$, $Z = (Z_{ij}) \in M(n, \mathbf{C})$ (see exercise 2.10).

DEFINITION (4.14). A Lie group $G$ is said to **act** on a $C^\infty$-manifold $M$ if there is a $C^\infty$-mapping $G \times M \ni (x, p) \mapsto x \cdot p \in M$ such that

(i) $(xy) \cdot p = x \cdot (y \cdot p)$, $x, y \in G$, $p \in M$,

(ii) for all $x \in G$, the mapping $M \ni p \mapsto x \cdot p \in M$ is a $C^\infty$-diffeomorphism of $M$ onto itself.

By (i), $e \cdot p = p$, $p \in M$. We say $G$ acts **effectively** on $M$ if for each $x \in G$, the condition that $x \cdot p = p$, for all $p \in M$ implies that $x = e$. We say that $G$ acts **transitively** on $M$ if for all $p, q \in M$, there exists an element $x \in G$ such that $q = x \cdot p$.

(4.15) If $G$ acts transitively on $M$, we fix $p \in M$ and let $K := \{x \in G;\ x \cdot p = p\}$ which is a closed Lie subgroup of $G$. The quotient space $G/K$ is $C^\infty$-diffeomorphic onto $M$ by $G/K \ni xK \mapsto x \cdot p \in M$. We call $K$ the **isotropy subgroup** of $G$ at $p \in M$.

In the following, we assume $G$ is a compact Lie group and $K$ is a closed Lie subgroup of $G$. Then a Riemannian metric $g$ on the quotient space $G/K$ is **$G$-invariant** if for all $x \in G$,

$$\tau_x^* g = g.$$

When $K = \{e\}$, we say the Riemannian metric $g$ is **left invariant** (resp. **right invariant**) if $L_x^* g = g$, for all $x \in G$ (resp., $R_x^* g = g$, for all $x \in G$). We call a metric $g$ **bi-invariant** if it is both left and right invariant.

Given an arbitrary inner product $( , )$ on $\mathfrak{g}$. Then we can define a left invariant Riemannian metric $g$ on $G$ by

$$g_x(X_x, Y_x) = \langle X, Y \rangle, \qquad X, Y \in \mathfrak{g}, \ x \in G, \tag{4.16}$$

where $X_x, Y_x \in T_x G$, $x \in G$. In particular, if $\langle , \rangle$ is Ad $(G)$-invariant, i.e.,

$$\langle \mathrm{Ad}(x)X, \mathrm{Ad}(x)Y \rangle = \langle X, Y \rangle, \qquad X, Y \in \mathfrak{g}, \ x \in G, \tag{4.17}$$

then the metric $g$ on $G$ defined by (4.16) is bi-invariant. Any compact Lie group admits a bi-invariant Riemannian metric. For instance, if $G = \mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, then the inner products on $\mathfrak{g} = \mathfrak{so}(n)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$ satisfying (4.17) are given by

$$\langle X, Y \rangle = -\mathrm{tr}(XY), \qquad X, Y \in \mathfrak{g}.$$

Any $G$-invariant Riemannian metric on $G/K$ can be given as follows: First, we fix an inner product $\langle , \rangle_0$ on $\mathfrak{g}$ satisfying (4.17) and then let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle , \rangle$, i.e.,

$$\mathfrak{m} := \{X \in \mathfrak{g}; \langle X, Y \rangle_0 = 0, \ \text{for all } Y \in \mathfrak{k}\}.$$

Then the subspace $\mathfrak{m}$ satisfies

$$\begin{cases} \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} & \text{(direct sum)}, \\ \mathrm{Ad}(k)X \in \mathfrak{m} & \text{for all } X \in \mathfrak{m}, \ k \in K. \end{cases}$$

Then each $X \in \mathfrak{m}$ can be identified with a tangent vector $X_o \in T_o(G/K)$ at the origin $o = \{K\} \in G/K$ which is the tangent vector of a curve $t \mapsto \exp(tX) \cdot o \in G/K$ at $t = 0$:

$$X_o := \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot o.$$

Then this linear mapping $\mathfrak{m} \ni X \mapsto X_o \in T_o(G/K)$ is bijective. Now we take an inner product $\langle , \rangle$ on $\mathfrak{m}$ satisfying

$$\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle = \langle X, Y \rangle, \qquad k \in K, \ X, Y \in \mathfrak{m}. \tag{4.19}$$

Then we can define a $G$-invariant Riemannian metric $g$ on $G/K$ by

$$g_{xK}(\tau_{x*} X_o, \tau_{x*} Y_o) = \langle X, Y \rangle, \quad X, Y \in \mathfrak{m}, \ x \in G, \tag{4.20}$$

where $\tau_{x*}X_o$, $\tau_{x*}Y_o \in T_{xK}(G/K)$ are the images of $X_o$, $Y_o \in T_o(G/K)$ by the differentiation of $\tau_x$. Since $xK = xkK$ for $k \in K$, to show the well definedness of $g$ by (4.20), it suffices to show that

$$g_{xkK}(\tau_{xk*}X_o, \tau_{xk*}Y_o) = g_{xK}(\tau_{x*}X_o, \tau_{x*}Y_o), \qquad x \in G,\ k \in K. \tag{4.21}$$

Note that $\tau_{xk*} = \tau_{x*}\tau_{k*}$ and the differentiation $\tau_{k*}$ of $\tau_k : G/K \to G/K$ maps $T_o(G/K)$ into itself. Moreover,

$$\tau_{k*}X_o = (\mathrm{Ad}(k)X)_o, \qquad X \in \mathfrak{m},\ k \in K,$$

because for $f \in C^\infty(G/K)$, we have

$$\begin{aligned}\tau_{k*}X_o(f) = X_o(f \circ \tau_k) &= \left.\frac{d}{dt}\right|_{t=0} f(k\exp(tX)\cdot o)\\ &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(t\mathrm{Ad}(k)X)\cdot o) \quad \text{(by (4.10))}\\ &= (\mathrm{Ad}(k)X)_o f.\end{aligned}$$

Thus, to obtain (4.21) we need that $\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y\rangle = \langle X, Y\rangle$ for all $k \in K$, $X$, $Y \in \mathfrak{m}$ which is (4.19).

Given a $G$-invariant metric $g$ on $G/K$, we call $(G/K, g)$ a **Riemannian homogeneous space** and $G/K$ a **homogeneous space**.

DEFINITION (4.22). For two Riemannian manifolds $(M, g)$, $(N, h)$, a $C^\infty$-mapping $\phi : (M, g) \to (N, h)$ is called an **isometry** if $\phi$ is a $C^\infty$-diffeomorphism of $M$ onto $N$ and satisfies $g = \phi^* h$. The **isometry group** Iso$(M, g)$ is by definition the totality of all isometries of $(M, g)$ into itself. This is a Lie group by the following multiplications:

$$\mathrm{Iso}(M, g) \times \mathrm{Iso}(M, g) \ni (\phi, \psi) \mapsto \phi \circ \psi \in \mathrm{Iso}(M, g)$$

is the composition map,

$$\mathrm{Iso}(M, g) \ni \phi \mapsto \phi^{-1} \in \mathrm{Iso}(M, g),$$

is the inverse map, and it acts on $M$ by

$$\mathrm{Iso}(M, g) \times M \ni (\phi, p) \mapsto \phi(p) \in M \quad \text{(see [K.N])}.$$

Note that

$$\{\tau_x ;\, x \in G\} \subset \mathrm{Iso}(G/K, g)$$

if $(G/K, g)$ is a Riemannian homogeneous space.

(4.23) EXAMPLE 3. Let us denote

$$\begin{aligned}\mathbf{R}^{n+1} &:= \{p = {}^t(p_1, \dots, p_{n+1});\, p_i \in \mathbf{R}\,(1 \le i \le n+1)\},\\ S^n &:= \{p \in \mathbf{R}^{n+1};\, \|p\| = 1\}.\end{aligned}$$

Then $\mathrm{SO}(n+1)$ acts on $S^n$ by multiplication of matrices and column vectors as

$$\mathrm{SO}(n+1) \times S^n \ni (x, p) \mapsto x \cdot p \in S^n.$$

We write $o = {}^t(1, 0, \dots, 0)$. Then

$$x \cdot o = {}^t(x_{11}, \dots, x_{n+1\,1}) \quad \text{(the first column vector of } x\text{)}.$$

$\mathrm{SO}(n+1)$ acts transitively on $S^n$, i.e.,

$$\text{for each } p \in S^n, \text{ there exists } x \in \mathrm{SO}(n+1), \qquad p = x \cdot o.$$

Because $n+1$ column vectors of an orthogonal matrix of degree $n+1$ give an orthonormal basis of $\mathbf{R}^{n+1}$ and any orthogonal matrix can be given by this way.

The isotropy subgroup $K$ of $G = \mathrm{SO}(n+1)$ at $o$ is given by

$$K = \left\{ \begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{0} & x \end{pmatrix} ; x \in \mathrm{SO}(n) \right\} \cong \mathrm{SO}(n),$$

where $\mathbf{0} = {}^t(0, \dots, 0)$. Thus, the unit sphere $S^n$ is expessed as

$$S^n = G/K = \mathrm{SO}(n+1)/\mathrm{SO}(n).$$

The Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ of $G$, $K$ are

$$\mathfrak{g} = \mathfrak{so}(n+1) \supset \mathfrak{k} = \left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} \in \mathfrak{so}(n+1) ; X \in \mathfrak{so}(n) \right\},$$

and the orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the inner product $\langle X, Y \rangle = -\mathrm{tr}(XY)$, $X, Y \in \mathfrak{g} = \mathfrak{so}(n+1)$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^tX_1 \\ X_1 & O \end{pmatrix} ; X_1 = {}^t(x_1, \dots, x_n) \in \mathbf{R}^n \right\}.$$

We give the inner product $\langle\, , \rangle_0$ by

$$\langle X, Y \rangle_0 := -\frac{1}{2} \mathrm{tr}(XY), \qquad X, Y \in \mathfrak{m}$$

which satisfies (4.19). The corresponding $G = \mathrm{SO}(n+1)$-invariant Riemannian metric $g_1$ on $G/K = \mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n$ coincides with the one $g_{S^n} = \iota^* g_0$. Because $g_{S^n}$ is also $SO(n+1)$-invariant, the corresponding inner product on $\mathfrak{m}$ is of the form $c\langle\, , \rangle_0$, $c > 0$, and the constant $c = 1$. In fact, for

$$Z := \begin{pmatrix} 0 & -e_1 \\ {}^te_1 & O \end{pmatrix} \in \mathfrak{m},$$

where $e_1 = {}^t(1, 0, \dots, 0)$, the geodesic

$$\sigma(t) = \exp(tZ) \cdot o = {}^t(\cos t, \sin t, 0, \dots, 0) \in S^n, \qquad -\infty < t < \infty$$

satisfies

$$g_{S^n}(\sigma'(0), \sigma'(0)) = \langle Z, Z \rangle_0 = 1.$$

REMARK. Recently, there has been a breakthrough in the development of the theory of homogeneous spaces. One of the fundamental problems in the theory is the following due to Hsiang-Lawson: Let $G$ be a compact Lie group and let $K$, $K'$ be its two closed subgroups. Does homeomorphicity mean diffeomorphicity for $G/K$ and $G/K'$ ?

But M. Kreck and S. Stolz [K.S] found the following counter example. Let $G = \mathrm{SU}(3)$, and $k$, $\ell$ integers mutually prime. Then

$$T_{k,\ell} := \left\{ \begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{i\ell\theta} & 0 \\ 0 & 0 & e^{-i(k+\ell)\theta} \end{pmatrix} ; \theta \in \mathbf{R} \right\},$$

is a one-dimensional closed Lie subgroup of $G$ and they showed that among the family of 7-dimensional homogeneous spaces $\{G/T_{k,\ell}; k, \ell\}$, there are distinct pairs $(k, \ell)$, $(k', \ell')$ such that (i) $G/T_{k,\ell}$ and $G/T_{k',\ell'}$ are homeomorphic, but (ii) $G/T_{k,l}$ and $G/T_{k',\ell'}$ are not diffeomorphic. They claim such pairs are

$$(k, \ell) = (-56788, 5227), \qquad (k', \ell') = (-42652, 61213).$$

The family of the above homogeneous space $G/T_{k,\ell}$ is famous: For all $(k, \ell)$, (i) $G/T_{k,\ell}$ admits a positively curved $G$-invariant Riemannian metric (cf. [A.W]). (ii) $G/T_{k,\ell}$ admits a $G$-invariant Einstein metric (cf. [W]). (iii) The spectrum of the Laplacian of the metric in (i) is determined (cf. [Ur2], [Ur4]). Kreck-Stolz's result implies that there exist two positively curved Riemannian manifolds which are homeomorphic but not diffeomorphic. It is very suprising that such a phenomena occurs among such rather simple homogeneous spaces.

**4.3. Infinite dimensional manifolds.** In this subsection, we assume $(M, g)$, $(N, h)$ are $m$, $n$ dimensional compact $C^\infty$-manifolds, and $(N, h)$ satisfies:

(i) $N$ is a closed submanifold of the Euclidean space $\mathbf{R}^K$, and if $\iota : N \subset \mathbf{R}^K$ denotes the inclusion, then

(ii) $h = \iota^* g_0$, where $g_0$ is the standard metric of $\mathbf{R}^K$.

By J.Nash's theorem, we may always assume $(N, h)$ satisfies the above.

In 1958, J.Eells [E] showed that the totality of smooth maps of $M$ into $N$ can be regarded as an infinite dimensional smooth manifold, and this study was developed into the study of harmonic mappings, one of the themes of this book. It is also related to infinite dimensional Lie group theory (see [Om]). We prepare some materials in analysis in order to state Eells' theorem for more precisely.

(4.24) For $k = 0, 1, \dots, \infty$, let $C^k(M, \mathbf{R}^K)$ be the totality of all $C^k$-mappings of $M$ into $\mathbf{R}^K$. We denote elements in $C^k(M, \mathbf{R}^K)$ by

$$u = (u_1, \dots, u_K) \in C^k(M, \mathbf{R}^K),$$

where $u_A \in C^k(M, \mathbf{R}) = C^k(M)$, $1 \le A \le K$. For $k \ge 1$, we put

$$du := (du_1, \dots, du_K), \tag{4.25}$$

where $du_A$, $1 \leq A \leq K$, are $C^{k-1}$-1-forms on $M$. At each point $x$ in $M$, we take the following norms $|\cdot|_x$, $|\cdot|$:

$$|du|_x := \left(\sum_{A=1}^{K} |du_A|_x^2\right)^{1/2}, \tag{4.26}$$

$$|u(x)| := \left(\sum_{A=1}^{K} u_A(x)^2\right)^{1/2}, \qquad x \in M,$$

where the norm $|\cdot|_x$ in the right-hand side of the first equality is the one on $T_x^*M$ induced from the Riemannian metric $g$ on $M$. We often abbreviate $x$ as $|du|$, $|u|$. Moreover, for $1 < p < \infty$, we define the $\|\cdot\|_{1,p}$ on $C^\infty(M, \mathbf{R}^K)$ by

$$\|u\|_{1,p} := \left(\int_M |du|^p v_g + \int_M |u|^p v_g\right)^{1/p}, \qquad u \in C^\infty(M, \mathbf{R}^K). \tag{4.27}$$

We denote by $L_{1,p}(M, \mathbf{R}^K)$ the Banach space completion of $C^\infty(M, \mathbf{R}^K)$ with respect to the norm $\|\cdot\|_{1,p}$. We also denote by $L_p(M, \mathbf{R}^K)$ the Banach space completion of $C^\infty(M, \mathbf{R}^K)$ with respect to the norm $\|\cdot\|_p$ defined by

$$\|u\|_p := \left(\int_M |u|^p v_g\right)^{1/p}, \qquad u \in C^\infty(M, \mathbf{R}^K).$$

We also define the following norm on the totality $C^0(M, \mathbf{R}^K)$ of all continuous mappings of $M$ into $\mathbf{R}^K$:

$$\|u\|_\infty := \sup\{|u(x|; x \in M\}. \tag{4.29}$$

It is known (see [Sm], [Lw.M]) that

SOBOLEV'S LEMMA (4.30). (i) *If* $1 > \frac{m}{p}$, $m = \dim(M)$, *then we have*

$$L_{1,p}(M, \mathbf{R}^K) \subset C^0(M, \mathbf{R}^K),$$

*and the inclusion is completly continuous.*

(ii) *If* $k - \frac{m}{p} > \ell$, $k, \ell \geq 0$, $k$ *and* $\ell$ *are integers, then*

$$L_{k,p}(M, \mathbf{R}^K) \subset C^\ell(M, \mathbf{R}^K),$$

*and the inclusion is completely continuous.*

(iii) *If* $k - \frac{m}{p} \geq \ell - \frac{m}{q}$, $k \geq \ell$, $k, \ell \in \mathbf{R}$, *then*

$$L_{k,p}(M, \mathbf{R}^K) \subset L_{\ell,q}(M, \mathbf{R}^K),$$

*and the inclusion is continuous. In particular, if* $k - \frac{m}{p} > \ell - \frac{m}{q}$ *and* $k > \ell$, *then the inclusion is completely continuous.*

Here the inclusion $L_{1,p}(M, \mathbf{R}^K) \subset C^0(M, \mathbf{R}^K)$ is said to be **completely continuous** if whenever $\{\phi_i\}_{i=1}^\infty$ is a bounded sequence in $L_{1,p}(M, \mathbf{R}^K)$; that is, there exists a positive constant $C > 0$ such that

$$\|\phi_i\|_{1,p} \leq C \quad \text{for all} \quad i = 1, 2, \ldots,$$

then there exists a subsequence $\{\phi_{i_k}\}_{k=1}^{\infty}$ which is convergent in $C^0(M, \mathbf{R}^K)$, i.e.,

$$\|\phi_{i_k} - \phi\|_\infty \to 0 \quad \text{as} \to \infty$$

for some $\phi \in C^0(M, \mathbf{R}^K)$.

Now we explain Eells' theorem. In the following we always make the important assumption:

$$1 > \frac{p}{m}, \qquad m = \dim(M) \tag{4.31}$$

which implies that by (i) of Sobolev's Lemma (4.30), each element in $L_{1,p}(M, \mathbf{R}^K)$ can be regarded as a continuous mapping of $M$ into $\mathbf{R}^K$. Thus, we can define the space

$$L_{1,p}(M, N) := \{\phi \in L_{1,p}(M, \mathbf{R}^K); \phi(x) \in N, \ x \in M\} \subset C^0(M, N).$$

Then we obtain

THEOREM (4.33) (J. Eells [E], 1958). *Assume that* $1 > \frac{m}{p}$, $m = \dim(M)$. *Then* $L_{1,p}(M, N)$ *is an infinite dimensional* $C^\infty$*-manifold, and the tangent space at* $\phi \in L_{1,p}(M, N)$, $T_\phi L_{1,\phi}(M, N)$, *is*

$$L_{1,p}(\phi^{-1}TN) := \{X \in L_{1,p}(M, \mathbf{R}^K); X(x) \in N_{\phi(x)}, \ x \in M\}. \tag{4.34}$$

Before giving a proof of this theorem, we explain its meaning.

(4.35) *The meaning of Theorem* (4.33). Note that for each point $y \in \mathbf{R}^K$, the tangent space $T_y\mathbf{R}^K$ is identified with $\mathbf{R}^K$ itself. The differentiation of the inclusion map $\iota : N \subset \mathbf{R}^K$ at $y \in N$,

$$d\iota : T_yN \to T_y\mathbf{R}^K \cong \mathbf{R}^K$$

is injective. We denote by $N_y$, the subspace $d\iota(T_yN)$ of $\mathbf{R}^K$. Any $C^0$-section of $\phi^{-1}TN$ is a continuous mapping $X : M \to TN$ satisfying

$$X(x) \in T_{\phi(x)}N, \qquad x \in M.$$

By the above, $X$ can be also regarded as a continuous mapping $X : M \to \mathbf{R}^K$ satisfying

$$X(x) \in N_{\phi(x)}, \qquad x \in M. \tag{4.36}$$

Therefore, together with Sobolev's lemma (4.30), the left-hand side of (4.34) can be regarded as the intersection of $L_{1,p}(M, \mathbf{R}^K)$ and the totality of all continuous mapping of $M$ into $\mathbf{R}^K$ satisfying $X(x) \in N_{\phi(x)}$ for all $x \in M$.

(4.37) $L_{1,p}(\phi^{-1}TN)$ is a closed subspace of the Banach space $L_{1,p}(M, \mathbf{R}^K)$, and then it is itself a Banach space.

In fact, for $X, Y \in L_{1,p}(\phi^{-1}TN)$, $\lambda, \mu \in \mathbf{R}$, let $(\lambda X + \mu Y)(x) := \lambda X(x)+\mu Y(x)$, $x \in M$. Then $\lambda X+\mu Y \in L_{1,p}(M, \mathbf{R}^K)$, and for all $x \in M$, $(\lambda X+\mu Y)(x) \in N_{\phi(x)}$ since $X(x), Y(x) \in N_{\phi(x)}$. Furthermore, if a sequence $\{X_i\}_{i=1}^{\infty}$ in $L_{1,p}(\phi^{-1}TN)$ converges to $X \in L_{1,p}(M, \mathbf{R}^K)$ with respect to the norm $\| \cdot \|_{1,p}$, then by Sobolev's lemma (4.30), it converges with respect to $\| \cdot \|_{\infty}$. Therefore, for each $x \in M$, $X_i(x) \to X(x)$ as $i \to \infty$. Since $X_i(x) \in N_{\phi(x)}$, which is a closed subspace of $\mathbf{R}^K$, $X(x) \in N_{\phi(x)}$. Thus, $X \in L_{1,p}(\phi^{-1}TN)$. □

(4.38) We explain intuitively that $L_{1,p}(\phi^{-1}TN)$ is regarded as the tangent space of $L_{1,p}(M, N)$ at $\phi$.

In general, a tangent vector of a manifold $M$ at $p$ is by definition the tangent vector $c'(0)$ of a $C^1$-curve $c(t)$ through $p$, i.e., $c(0) = p$ (cf. (2.2)).

At a point $\phi$ of $L_{1,p}(M,N)$ we may consider a $C^1$-curve in $L_{1,p}(M,N)$, $c_\phi : I \ni t \mapsto c_\phi(t) \in L_{1,p}(M, N)$, with $c_\phi(0) = \phi$. Here $I$ is an open interval containing $0$. That is, $\{c_\phi(t)\}_{t\in I}$ is a one parameter family of $C^1$-mappings of $M$ into $N$ satisfying $c_\phi(0) = \phi$:

$$\begin{cases} c_\phi(t) : M \ni x \mapsto c_\phi(t)(x) \in N \subset \mathbf{R}^K, \\ c_\phi(0) = \phi; \quad \text{that is, } c_\phi(0)(x) = \phi(x), \ x \in M. \end{cases}$$

Then the **tangent vector** of $c_\phi(t)$ at $t = 0$ is

$$X := \left.\frac{d}{dt}\right|_{t=0} c_\phi(t).$$

(See Figure 2.13.) This means that for $x \in M$,

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} c_\phi(t)(x)$$

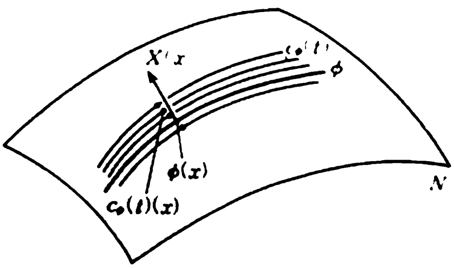



FIGURE 2.13

and $I \ni t \mapsto c_\phi(t)(x) \in N$ is a $C^1$-curve in $N$ through $\phi(x)$ at $t = 0$. Thus, since

$$X(x) = \frac{d}{dt}\bigg|_{t=0} c_\phi(t)(x) \in T_{\phi(x)}N \cong N_{\phi(x)} \subset \mathbf{R}^K,$$

it follows that

$$X : M \to \mathbf{R}^K, \quad \text{is a continuous mapping with } X(x) \in N_{\phi(x)}, \quad x \in M. \tag{4.39}$$

Moreover, since $t \mapsto c_\phi(t)$ is a $C^1$-curve in the Banach space $L_{1,p}(M, \mathbf{R}^K)$, $X = \frac{d}{dt}\big|_{t=0} c_\phi(t) \in L_{1,p}(M, \mathbf{R}^K)$. Therefore, together with (4.39), we get $X \in L_{1,p}(\phi^{-1}TN)$.

Thus, we adopt $L_{1,p}(\phi^{-1}TN)$ for the tangent space $T_\phi L_{1,p}(M, N)$ at $\phi \in L_{1,p}(M, N)$, so in order to introduce a $C^\infty$-structure to $L_{1,p}(M, N)$, it suffices to show that $L_{1,p}(M, N)$ is a manifold modelled by a Banach space $L_{1,p}(\phi^{-1}TN)$.

REMARK. The definition of a manifold in subsection 2.1 is the one to be modelled to a fixed Banach space $E$. In order to show how to equip $L_{1,p}(M, N)$ with a manifold structure, we should redefine the notion of a manifold as follows (then several notions in §2 are given in a similar way).

DEFINITION (4.40). A Hausdorff space $M$ is a **manifold** if for each point $p \in M$, there exists an open neighborhood $U_\alpha$ in $M$ containing $p$, and a diffeomorphism $\alpha$ of $U_\alpha$ onto an open subset $\alpha(U_\alpha)$ of a Banach space $E_\alpha$. Moreover, $M$ is a **$C^k$-(Banach) manifold** if

(i) $M = \bigcup_{\alpha \in A} U_\alpha$,

(ii) for two $(U_{\alpha_1}, \alpha_1)$, $(U_{\alpha_2}, \alpha_2)$ with $U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset$, the mapping $\alpha_2 \circ \alpha_2^{-1} : E_{\alpha_1} \supset \alpha_1(U_{\alpha_1} \cap U_{\alpha_2}) \to \alpha_2(U_{\alpha_1} \cap U_{\alpha_2}) \subset E_{\alpha_2}$ is a $C^k$-diffeomorphism. We call also $(U_\alpha, \alpha)$ a **coordinate neighborhood** of $M$.

Now we start to prove Eells' theorem (4.33).

(4.41) *The first step in proving Theorem* (4.33). We give a coordinate neighborhood around $\phi \in L_{1,p}(M, N)$ as follows:

(i) First, we define the exponential mapping

$$\exp_\phi : T_\phi L_{1,p}(M, N) = L_{1,p}(\phi^{-1}TN) \to L_{1,p}(M, N).$$

Let exp be the exponential mapping of $(N, h)$. Then we define

$$\exp_\phi : X \mapsto \exp \circ X. \tag{4.42}$$

The circle in (4.42) denotes the composition of two mappings. For $X \in L_{1,p}(\phi^{-1}TN)$, by means of $X(x) \in N_{\phi(x)} \cong T_{\phi(x)}N$, $x \in M$, we get $(\exp \circ X)(x) := \exp_{\phi(x)} X(x) \in N$. Since $N$ is compact, by Hoph-Rinow's theorem (3.15), $(N, h)$ is complete, so we can define $\exp_{\phi(x)} X(x)$. Since $\phi \in L_{1,p}(M, N)$, and $X \in L_{1,p}(\phi^{-1}TN)$, we get $\exp \circ X \in L_{1,p}(M, N)$.

The mapping $\exp_\phi$ maps $0 \in L_{1,p}(\phi^{-1}TN)$ to $\phi$ and is the one onto a neighborhood of $\phi$ in $L_{1,p}(M, N)$.

Because if $\phi' \in L_{1,p}(M, N)$ is close to $\phi$ in the sense of the norm $\|\cdot\|_{1,p}$, then it is close in the sense of $\|\cdot\|_\infty$ by Sobolev's lemma (4.30). Note that any point $y'$ in a neighborhood of $y \in N$ can be written uniquely as $y' = \exp_y u$, $u \in T_yN$. Therefore, for any $x \in M$, there exists a unique $X(x) \in T_{\phi(x)}N$ such that

$$\phi'(x) = \exp_{\phi(x)} X(x).$$

Since $\phi' \in L_{1,p}(M, N)$, we get $X \in L_{1,p}(\phi^{-1}TN)$ and $\phi' = \exp \circ X$ by definition (4.42).

(ii) Using $\exp_\phi : T_\phi L_{1,p}(M, N) \to L_{1,p}(M, N)$, we can define a coordinate neighborhood of each ponit $\phi$ in $L_{1,p}(M, N)$ as follows. Take a sufficiently small neighborhood $V_\phi$ of 0 in $T_\phi L_{1,p}(M, N) = L_{1,p}(\phi^{-1}TN)$, put $U_\phi := \exp_\phi(V_\phi)$. Then we obtain a coordinate neighborhood of $\phi$

$$\exp_\phi^{-1} : U_\phi \to V_\phi \subset L_{1,p}(\phi^{-1}TN).$$

(4.43) *The second step in proving Theorem* (4.33). Choosing such two coordinate neighborhoods $U_\phi$, $U_{\phi'}$ with $U_\phi \cap U_{\phi'} \neq \emptyset$, for $\phi, \phi' \in L_{1,p}(M, N)$, it suffices to show

$$\Phi := \exp_{\phi'}^{-1} \circ \exp_\phi : L_{1,p}(\phi^{-1}TN) \supset V_\phi \to V_{\phi'} \subset L_{1,p}(\phi'^{-1}TN)$$

is a $C^\infty$-diffeomorphism. See Figure 2.14.

For $X \in V_\phi$, $Y = \Phi(X)$ is determined as $\exp_\phi(X) = \exp_{\phi'}(Y)$, i.e., $\exp_{\phi(x)} X(x) = \exp_{\phi'(x)} Y(x)$, $x \in M$. Since the diffeomorphisms of the neighborhoods around $\phi(x)$, $\phi'(x)$ into open sets in $\mathbf{R}^n$ are given by $\exp_{\phi(x)}^{-1}$, $\exp_{\phi'(x)}^{-1}$ and satisfy $Y(x) = (\exp_{\phi'(x)}^{-1} \circ \exp_{\phi(x)})X(x)$, the mapping $\Psi$ defined by

$$\Psi_x := \exp_{\phi'(x)}^{-1} \circ \exp_{\phi(x)}, \qquad x \in M$$

is a $C^\infty$-diffeomorphism by definition since $N$ is a $C^\infty$-manifold. Showing that $\Phi$ is a $C^\infty$ mapping of an open subset in $L_{1,p}(\phi^{-1}TN)$ onto an open subset in $L_{1,p}(\phi'^{-1}TN)$ reduces to proving the following lemma using the definition of $\Phi$, the inclusions $L_{1,p}(\phi^{-1}TN)$, $L_{1,p}(\phi'^{-1}TN) \subset L_{1,p}(M, \mathbf{R}^K)$ and the partition of unity on $M$ (cf. subsection 3.6). See Figure 2.15.

LEMMA (4.44). *Let* $1 > \frac{m}{p}$, $m = \dim(M)$. *Let* $V$ *be a coordinate neighborhood in* $M$, *and let* $U$ *be an open set in* $M$ *satisfying* $\overline{U} \subset V$. *Assume that the mapping*

$$\Psi : V \times \mathbf{R}^K \ni (x, \xi) \mapsto \Psi(x, \xi) \in \mathbf{R}^K$$

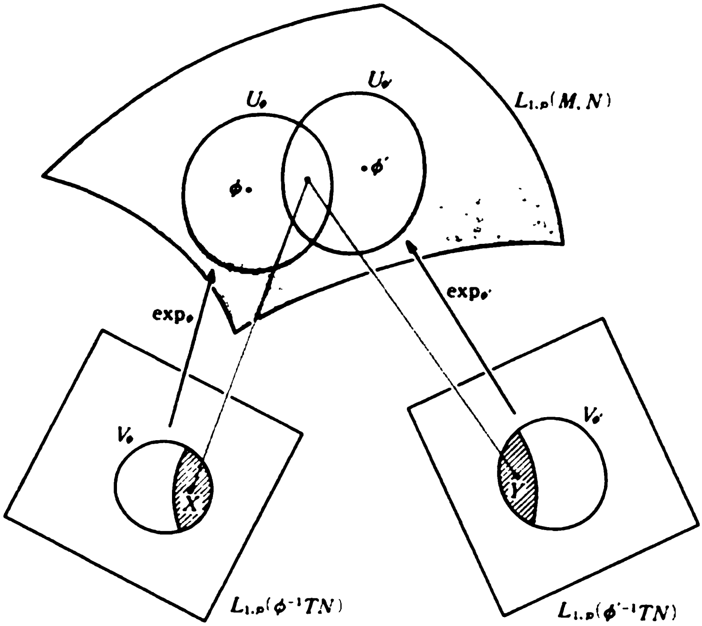



FIGURE 2.14

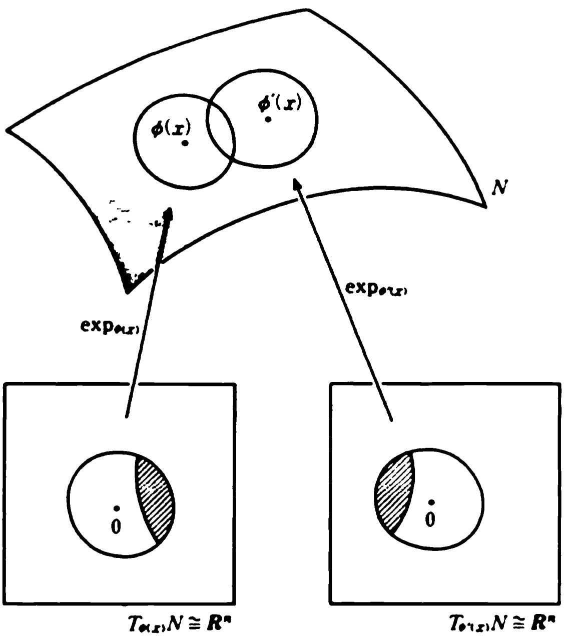



FIGURE 2.15

*satisfies the following conditions:*

(i) *For* $\xi \in \mathbf{R}^k$, *the mapping* $V \ni x \mapsto \Psi(x, \xi) \in \mathbf{R}^k$ *belongs to* $L_{1,p}(V, \mathbf{R}^K)$. *Denoting its norm by* $\|\Psi(\cdot, \xi)\|_{1,p}$, *then* $\xi \mapsto \|\Psi(\cdot, \xi)\|_{1,p}$ *is continuous.*

(ii) *For each* $x \in V$, *the mapping* $\mathbf{R}^K \ni \xi \mapsto \Psi(x, \xi) \in \mathbf{R}^K$ *is* $C^\infty$.

*Then it follows that:*

(1) *Defining* $\Psi \circ X(x) := \Psi(x, X(x))$, $x \in U$, *for* $X \in L_{1,p}(U, \mathbf{R}^K)$, $\Psi \circ X$ *belongs to* $L_{1,p}(U, \mathbf{R}^K)$.

(2) *The mapping* $\Phi : L_{1,p}(U, \mathbf{R}^K) \ni X \mapsto \Psi \circ X \in L_{1,p}(U, \mathbf{R}^K)$ *is a* $C^\infty$*-mapping of the Banach space* $L_{1,p}(U, \mathbf{R}^K)$ *into itself.*

Proof of the Lemma. (1) By means of the composition rule of differentiation, the differentiation of the mapping $x \mapsto \Psi(x, X(x))$ is

$$d_x\Psi(x, X(x))\frac{\partial \Psi}{\partial x}(x, X(x)) + \frac{\partial \Psi}{\partial \xi}(x, X(x))\, d_x X.$$

Therefore, we get

$$\begin{aligned}
&\left(\int_U |d_x\Psi(x, X(x))|^p\, dx\right)^{1/p} \\
&\leq \left(\int_U \left|\frac{\partial \Psi}{\partial x}(x, X(x))\right|^p dx + \int_U \left|\frac{\partial \Psi}{\partial \xi}(x, X(x))\right|^p |d_x X|^p\, dx\right)^{1/p} \\
&\leq \left(\sup_{\xi \in V} \int_U \left|\frac{\partial \Psi}{\partial x}(x, \xi)\right|^p dx\right)^{1/p} \\
&\quad + \left(\sup_{x \in U}\left|\frac{\partial \Psi}{\partial \xi}(x, X(x))\right|^p \int_U |d_x X|^p\, dx\right)^{1/p} < \infty.
\end{aligned}$$

Here we use the fact that $\{X(x) \in \mathbf{R}^K ; x \in U\}$ is bounded in $\mathbf{R}^K$. By a similar argument, we get $\int_U |\Psi(x, X(x))|^p\, dx < \infty$, so we obtain (1).

(2) If $t \mapsto X_t$ is a $C^\infty$-curve in $L_{1,p}(U, \mathbf{R}^K)$ satisfying $X_0 = X$ at $t = 0$, then $t \mapsto \Psi(\cdot, X_t(\cdot))$ is also a $C^\infty$-curve in $L_{1,p}(U, \mathbf{R}^K)$ since $\Psi(x, \xi)$ is $C^\infty$ in $\xi$. So using Zorn's proposition (1.12), we can show that $\Phi$ is $C^\infty$. □

Thus, we obtain Theorem (4.33). □

Before closing Chapter 2, we explain connectivity with regard to $C^0(M,N)$.

(4.45) *Connectivity about* $C^0(M, N)$ *and the* **homotopy**. Here we explain the connectivity about the set $C^0(M, N)$ of all continuous mappings of $M$ into $N$, briefly. See [Ml] for more detail. Two points $\phi$, $\psi$ in $C^0(M, N)$ are called **connected** if there exists a continuous curve $\phi_t : 0 \leq t \leq 1$ in $C^0(M, N)$ satisfying $\phi_0 = \phi$ and $\phi_1 = \psi$.

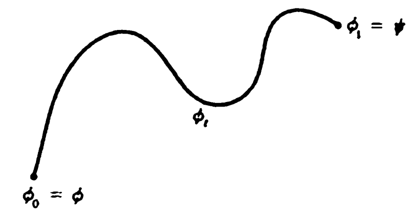



FIGURE 2.16

That is, putting

$$F(t, x) := \phi_t(x), \qquad x \in M, \ t \in I = [0, 1],$$

$F$ satisfies

(1) $F : I \times M \to N$ is continuous,
(2) $F(0, x) = \phi(x)$, $x \in M$,
(3) $F(1, x) = \psi(x)$, $x \in M$.

In this case, the continuous mappings $\phi$, $\psi$ are called to be **homotopic** and we write $\phi \sim \psi$. See Figure 2.16. This is an equivalence relation:
(i) $\phi \sim \phi$,
(ii) $\phi \sim \psi$ implies $\psi \sim \phi$,
(iii) $\phi \sim \psi$, and $\psi \sim \eta$ implies $\phi \sim \eta$.

We call the set of all equivalence classes $[\phi]$, $\phi \in C^0(M, N)$, the **free homotopy**, denoted by $[M, N]$. In particular, if $M = S^m$ ($m$-dimensional sphere), then $[S^m, N]$ admits a natural group structure. This group is called the **$m$-th homotopy group** of $N$, denoted by $\pi_m(N)$. $\pi_1(N)$ is called simply the **fundamental group** of $N$. If $\pi_1(N) = \{0\}$, then $N$ is called **simply connected**. Since $S^0 = \{-1, 1\}$, the number of all elements of $\pi_0(N)$ coincides with the number of connected components of $N$. In general, $[M, N]$ does not consist of just one element and then $C^0(M, N)$ is not connected.

**Exercises**

2.1. Let $E$, $F$ be two Banach spaces, let $B : E \times E \to F$ be a continuous bilinear mapping. Define a mapping $f$ of $E$ into $F$ by $f(x) = B(x, x)$, $x \in E$. Then prove the following:
 (i) $df_p(x) = 2B(p, x)$, $p$, $x \in E$,
 (ii) $d^2 f_p(x, y) = 2B(x, y)$, $p$, $x$, $y \in E$,
 (iii) $d^3 f_p = 0$, $p \in E$, thus $f$ is a $C^\infty$-mapping.

2.2. In the totality $L(E, F)$ of all bounded linear mappings of a Banach space $E$ into another one $F$, define addition, scalar multiplication, and norm by

$$(\lambda T_1 + \mu T_2)(x) := \lambda T_1(x) + \mu T_2(x),$$
$$\lambda, \mu \in \mathbf{R},\ T_1,\ T_2 \in L(E, F),\ x \in E,$$
$$\|T\| := \sup\{\|T(x)\|/\|x\|;\ 0 \neq x \in E\}, \qquad T \in L(E, F).$$

Show $L(E, F)$ is a Banach space.

2.3. For an $n$-dimensional Riemannian manifold $(M, g)$, show that

$$\operatorname{div}(X) = \sum_{i=1}^{n} g(e_i, \nabla_{e_i} X) = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\sqrt{g} X_i),$$

for $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$. Here $\{e_i\}_{i=1}^{n}$ is an orthonormal frame field and $\sqrt{g} = \sqrt{\det(g_{ij})}$.

2.4. For $f,\ f_1,\ f_2 \in C^\infty(M)$, show that

(i) $\operatorname{grad} f = \sum_{i=1}^{n} e_i(f) e_i = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$,

(ii) $g(\operatorname{grad} f_1, \operatorname{grad} f_2) = g(df_1, df_2)$.

2.5. For $f \in C^\infty(M)$, show that

$$\Delta f = -\operatorname{div}\operatorname{grad} f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x_j}\right)$$
$$= -\sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k}\right) = \sum_{i=1}^{n}(e_i{}^2 f - \nabla_{e_i} e_i f).$$

2.6. Show that

$$\delta\eta = -\sum_{i=1}^{n} (\nabla_{e_i} \eta)(e_i) = -\operatorname{div}(X), \qquad \eta \in A^1(M),$$

where $X \in \mathfrak{X}(M)$ is determined by $g(X, Y) = \eta(Y)$, $Y \in \mathfrak{X}(M)$.

2.7. Show that

$$d\omega(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, \ldots, \widehat{X}_i, \ldots, X_{r+1}),$$

for $\omega \in A^r(M)$, $X_1, \ldots, X_{r+1} \in \mathfrak{X}(M)$.

2.8. Let $\{v_i\}_{i=1}^{n}$ be a basis for $\mathbf{R}^n$, and let $\Lambda = \{\sum_{i=1}^{n} m_i v_i;\ m_i \in \mathbf{Z},\ 1 \leq i \leq n\}$. Show that the Riemannian metric $g_\Lambda$ on $\mathbf{R}^n/\Lambda$ induced from the standard one $g_0$ on $\mathbf{R}^n$ has the form

$$g_\Lambda = \sum_{i,j=1}^{n} g_{ij}\, dy_i \otimes dy_j.$$

Here $g_{ij} = (v_i, v_j)$ where $(\ ,\ )$ is the standard inner product, and $(y_1, \ldots, y_n)$ is the local coordinate given by $\mathbf{R}^n/\Lambda \ni \pi(\sum_{i=1}^{n} y_i v_i) \ni\!\mapsto (y_1, \ldots, y_n)$.

2.9. Show the vector field (4.12) $\tilde{X}$ in Example 1 (4.11) is left invariant on $\mathrm{GL}(n, \mathbf{R})$, and the equations

$$[\tilde{X}, \tilde{Y}] = [X, Y]^{\sim}, \qquad \exp(\tilde{X}) = e^X,$$

for $X, Y \in \mathfrak{gl}(n, \mathbf{R})$.

2.10. In (4.13) Example 2, show that $\mathrm{O}(n)$, $\mathrm{SO}(n)$ are compact closed Lie subgroups of $\mathrm{GL}(n, \mathbf{R})$, and $\mathrm{U}(n)$, $\mathrm{SU}(n)$ are compact Lie groups. Moreover, show that $\mathfrak{so}(n)$ is Lie algebra of $\mathrm{O}(n)$, $\mathrm{SO}(n)$ and that $\mathfrak{u}(n)$, $\mathfrak{su}(n)$ are Lie algebras of $\mathrm{U}(n)$, $\mathrm{SU}(n)$, respectively.

CHAPTER 3

# Morse Theory

To study the topology of the manifold, M. Morse developed the theory, called Morse theory, of minima and maxima of a function on a manifold. In the early 1960's, R. Palais and S. Smale studied the theories of critical points of a function on an infinite dimensional manifold to apply the variational method. They clarified that the socalled condition (C) is necessary for a given function to admit a minimum. However, to satisfy the condition (C), we need the borderline estimate of Sobolev's lemma. We explain this theory in this chapter.

Unfortunately, many interesting variational problems do not satisfy the condition (C). To overcome this difficulty is one of the main problems in the fields of analysis and geometry. We shall show one of the methods to solve it in Chapter 6. This method is due to K. Uhlenbeck.

## §1. Critical points of a smooth function

**1.1. Introduction.** We start with the study of the behavior of a given function on a neighborhood of the origin in the Euclidean space near its critical point. For examples, we consider the following functions on the $xy$-plane:

(i) $f_1(x, y) = x^2 + y^2$,
(ii) $f_2(x, y) = -x^2 + y^2$,
(iii) $f_3(x, y) = -x^2 - y^2$,
(iv) $f_4(x, y) = y^2$.

For these functions, one can see easily their graph by their form, and we know their behavior around the $(x, y) = (0, 0)$. (See Figure 3.1, next page.)

As above, if a given function on $\mathbf{R}^n$ around the origin $\mathbf{0}$ is written in the following form (called the **canonical form**)

$$f(x) = f(x_1, \dots, x_n) = f(\mathbf{0}) - {x_1}^2 - \cdots - {x_\lambda}^2 + {x_{\lambda+1}}^2 + \cdots + {x_n}^2, \quad (1.1)$$

then one can see the behavior of a function around the origin. The following Morse lemma guarantees this.

**Morse Lemma (1.2).** *Assume that a given smooth function* $f(x) = f(x_1, \ldots, x_n)$ *on a neighborhood of a point* $p = (p_i, \ldots, p_n)$ *of* $\mathbf{R}^n$ *satisfies*

$$\text{(i)} \quad \frac{\partial}{\partial x_1}(p) = \cdots = \frac{\partial}{\partial x_n}(p) = 0,$$

*and*

$$\text{(ii)} \quad \textit{the } n \times n \textit{ matrix} \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right) \textit{ is nonsingular},$$

*that is, the determinant is not zero. Then by changing the variable* $y_i = y_i(x_1, \ldots, x_n)$, $1 \le i \le n$, *where* $y_i(p_1, \ldots, p_n) = 0$, $1 \le i \le n$, *we can reduce to*

$$f(y_1, \cdots, y_n) = f(p) - {y_1}^2 - \cdots - {y_\lambda}^2 + {y_{\lambda+1}}^2 + \cdots + {y_n}^2. \tag{1.3}$$

Here a point $p$ is a **critical point** of $f$ if it satisfies (i) of the Morse Lemma (1.2). The matrix in (ii) in (1.2) is called the **Hessian** of $f$ at the critical point $p$, and the number $\lambda$, i.e., the number of the negative terms $-{y_i}^2$, $(1 \le i \le \lambda)$ is called the **index** which influences greatly to the graph

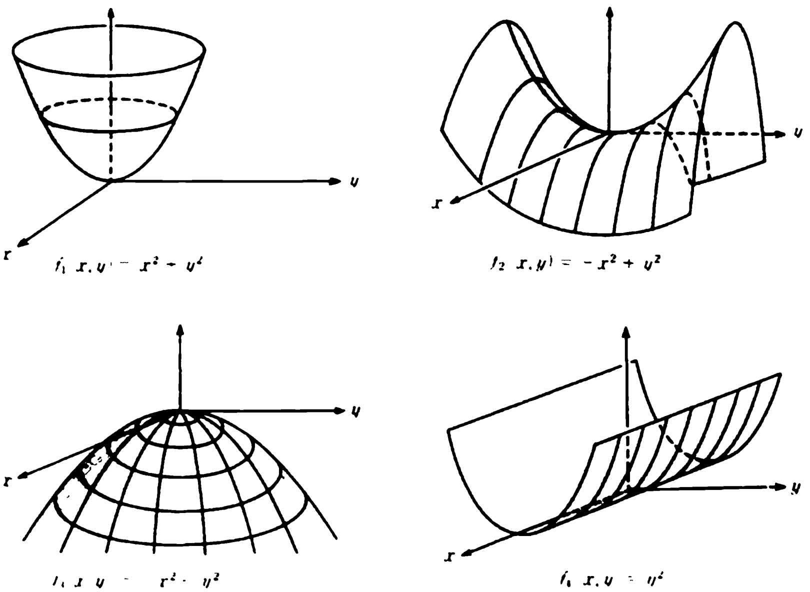


Figure 3.1

of $f$. The index coincides with the number of the negative eigenvalues of the Hessian. If (ii) in (1.2) is satisfied, then the critical point $p$ is called **nondegenerate**. It turns out that a nondegenerate critical point $p$ is isolated; there is no other critical point in some neighborhood of $p$. In general, a critical point might be degenerate like the function in (iv), and the Hessian might have zero as an eigenvalue. The multiplicity of the zero eigenvalue is called the **nullity** of the critical point.

Let $f$ be a differentiable function on a finite dimensional smooth manifold $M$. Then $p \in M$ is called a **critical point** of $f$ if it satisfies $df_p = 0$. This is equivalent to $u(f) = 0$, for each $u \in T_pM$. Studying this critical point of a smooth function on $M$ is very useful for studying the topology of $M$. Such a theory is called Morse theory. For instance, it is known (see [Ml] for more detail) that

THEOREM (1.4) (Reeb). *A compact n-dimensional $C^\infty$-manifold that admits a smooth function with only two nondegenerate critical points is homeomorphic to a sphere.*

**1.2. Critical points and regular points.** We start with a finite or infinite dimensional $C^k$-manifold as in §2 in Chapter 2.

Let $M$ be a $C^1$-manifold, and let $f : M \to \mathbb{R}$ be a $C^1$-function on $M$. Then by definition, for any point $p \in M$, the differentiation $df_p : T_pM \to \mathbb{R}$ is a bounded linear mapping. If $p \in M$ is a **critical point** of $f$ at $p$ if $df_p = 0$, that is, $df_p(u) = u(f) = 0$, for all $u \in T_pM$ (which implies, intuitively, the differentiations at $p$ in all directions vanish). Otherwise, $p$ is a **regular point**. For any $c \in \mathbb{R}$, $f^{-1}(c)$ is called a **level set** ( of height $c$ ). $f^{-1}(c)$ is called **regular** if any point in it is regular. Otherwise, $f^{-1}(c)$ is called **critical**. Moreover, $c$ is a **regular value** (resp., a **critical value**) if $f^{-1}(c)$ is regular (resp., critical). (See Figure 3.2.)

In the following, we always assume that $M$ is a $C^2$-manifold and $f$ is a $C^2$-function on $M$.

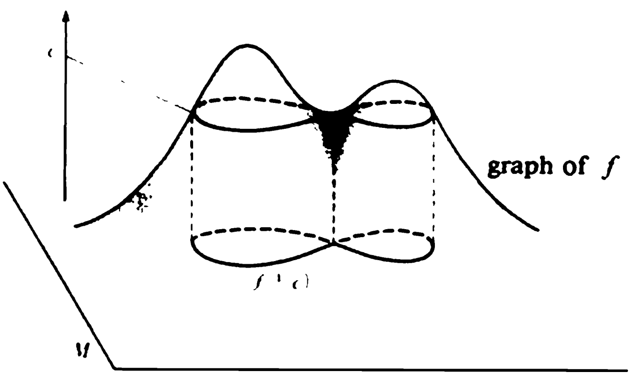



FIGURE 3.2

PROPOSITION (1.5). *Let $p \in M$ be a critical point of $f$. Then there exists a unique bounded symmetric bilinear mapping*

$$H(f)_p : T_pM \times T_pM \to \mathbf{R},$$

*such that for any coordinate neighborhood $(U_\alpha, \alpha)$, $\alpha : U_\alpha \to E$,*

$$H(f)_p(u, v) = d^2(f \circ \alpha^{-1})_{\alpha(p)}(d\alpha(u), d\alpha(v)), \qquad u, v \in T_pM, \quad (1.6)$$

*where $d\alpha(u)$, $d\alpha(v) \in T_{\alpha(p)}E = E$. $H(f)_p$ is called the* **Hessian** *of $f$ at $p$.*

PROOF. By definition, $f \circ \alpha^{-1}$ is a $C^2$-function on an open set $\alpha(U_\alpha)$ in a Banach space $E$ (cf. subsection 2.2 in Chapter 2) and its second differentiation with respect to the directions $d\alpha(u)$, $d\alpha(v)$ coincides with the right-hand side of (1.6). It suffices to show that $H(f)_p$ is independent of the choice of $(U_\alpha, \alpha)$. We need the following lemma:

LEMMA (1.7). *Let $U$, $U' \subset E$, be two open sets, and let $\varphi : U \to U'$ be an onto $C^k$-diffeomorphism ($k \geq 2$), let $f : U' \to \mathbf{R}$ be a $C^2$-function, and let $g := f \circ \varphi : U \to \mathbf{R}$. If $dg_p = 0$ at $p \in U$, then*

$$d^2 g_p(u, v) = d^2 f_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)), \qquad u, v \in E.$$

PROOF. For $x \in U$, and $u$, $v \in E$, we have that $dg_x = df_{\varphi(x)} \circ d\varphi_x$ and

$$d^2 g_x(u, v) = d^2 f_{\varphi(x)}(d\varphi_x(u), \varphi_x(v) + df_{\varphi(x)}(d^2\varphi_x(u, v)).$$

Letting $x = p$,

$$0 = dg_p = df_{\varphi(p)} \circ d\varphi_p.$$

Since $\varphi$ is a $C^k$-diffeomorphism, $d\varphi_p$ is a linear isomorphism, so that $df_{\varphi(p)} = 0$; that is, the second term of the above equation vanishes, which is the desired result. □

PROOF OF PROPOSITION (1.5) CONTINUED. We take two coordinate neighborhoods $\alpha : U_\alpha \to E$, $\beta : U_\beta \to E$. Put $\varphi := \alpha \circ \beta^{-1} : \beta(U_\alpha \cap U_\beta) \to \alpha(U_\alpha \cap U_\beta)$ which is a $C^2$-diffeomorphism. Then for $u$, $v \in T_pM$, we get

$$d\alpha(u) = d\varphi(d\beta(u)), \qquad d\alpha(v) = d\varphi(d\beta(v)).$$

Then we obtain

$$\begin{aligned} d^2(f \circ \alpha^{-1})_{\alpha(p)}(d\alpha(u), d\alpha(v)) &= d^2(f \circ \alpha^{-1})_{\alpha(p)}(d\varphi(d\beta(u)), d\varphi(d\beta(v)) \\ &= d^2(f \circ \alpha^{-1} \circ \varphi)_{\beta(p)}(d\beta(u), d\beta(v)) \quad \text{(by Lemma (1.7))} \\ &= d^2(f \circ \beta^{-1})_{\beta(p)}(d\beta(u), d\beta(v)). \qquad \square \end{aligned}$$

DEFINITION (1.8). A bounded symmetric bilinear mapping $B : E \times E \to \mathbf{R}$ on a Banach space $E$ is called **nondegenerate** if we can define a bounded linear mapping $T : E \ni v \mapsto T(v) \in E^*$ by

$$T(v)(w) := B(v, w), \qquad w \in E,$$

such that $T$ is a linear isomorphism of $E$ onto $E^*$. Otherwise, $T$ is called **degenerate.** The **index** is by definition

$$\sup\{\dim(W);\ B \text{ is negative definite on } W\},$$

where $W$ runs over all subspaces of $E$. Here $B$ is **negative definite** if

$$B(v, v) < 0, \quad \text{for all nonzero } v \in W.$$

The **nullity** of $B$ is

$$\dim\{v \in E;\ B(v, w) = 0, \text{ for all } w \in E\}.$$

Both the index and nullity may be infinite.

DEFINITION (1.9). Let $f : M \to \mathbf{R}$ be a $C^2$-function on a $C^2$-manifold $M$, and let $p \in M$ be a critical point of $f$. Then $p$ is called **degenerate** (resp., **nondegenerate**) if the Hessian $H(f)_p : T_pM \times T_pM \to \mathbf{R}$ is degenerate (resp., nondegenerate). The **index** (resp., **nullity**) is the index (resp., nullity) of the Hessian of $f$ at $p$. A critical point $p$ is **weakly stable** if the index of $f$ at $p$ is zero; that is,

$$H(f)_p(u, u) \geq 0, \quad \text{for each } u \in T_pM.$$

Note that the minimizer of $f$ is always weakly stable.

MORSE LEMMA (1.10). *Let $H$ be a Hilbert space, let $U$ be a convex neighborhood of* $0$ *in $H$, and let $f : U \to \mathbf{R}$ be a $C^{k+2}$-function, $k \geq 1$, with $f(0) = 0$. Assume that* $0$ *is a nondegenerate critical point of $f$. Then there exist an open neighborhood $V$ of* $0$ *contained in $U$ and a $C^k$-diffeomorphism $\varphi : V \to V$ of $V$ into itself with $\varphi(0) = 0$ and*

$$f(\varphi(v)) = \|Pv\| - \|(I - P)v\|^2, \qquad v \in V,$$

*where $P : H \to H$ is a projection* (see subsection 1.3).

REMARK. If $H = \mathbf{R}^n$, then (1.10) is (1.2).

**1.3. Spectral resolution of a selfadjoint operator.** We review basic facts about a selfadjoint operator on a Hilbert space which are necessary to explain the Morse Lemma (1.10). See Yosida [Y] for more detail.

(1.11) (Selfadjoint operators) The **adjoint operator** of a bounded operator $A : H \to H$ of a Hilbert space $(H, (\ ,\ ))$ is a bounded operator $A^* : H \to H$ satisfying

$$(Ax, y) = (x, A^*y), \qquad x, y \in H.$$

A bounded linear operator $A$ is called **selfadjoint** if $A = A^*$; that is,

$$(Ax, y) = (x, Ay), \qquad x, y \in H.$$

A selfadjoint operator $P : H \to H$ is called a **projection** if $P^2 = P$, where $P^2 := P \circ P$, the composition.

(1.12) (Resolution of the identity) A family $\{E(\lambda); -\infty < \lambda < \infty\}$ is called a **resolution of the identity** if it satisfies

(0) $E(\lambda)$ is a selfadjoint operator of $(H, \langle\ ,\ \rangle)$ and satisfies $E(\lambda)^2 = E(\lambda)$,

(i) $E(\lambda) \circ E(\mu) = E(\min\{\lambda, \mu\})$,

(ii) $E(-\infty) = 0$, $E(\infty) = I$, $\lim_{\lambda \to \mu+0} E(\lambda) = E(\mu)$.

Here $0$, $I$ are the **null operator** and the **identity operator**, respectively. $E(-\infty) := \lim_{\lambda \to -\infty} E(\lambda)$, $E(\infty) := \lim_{\lambda \to \infty} E(\lambda)$, and $\lim_{\lambda \to \mu+o} E(\lambda) = E(\mu)$ is **strong convergence**; that is,

$$\lim_{\lambda \downarrow \mu} \|(E(\lambda) - E(\mu))x\| = 0, \qquad x \in H.$$

By (i) of (1.12), the operator $E(\alpha, \beta] := E(\beta) - E(\alpha)$ is a projection. Moreover, by (i), (ii), for all $x, y \in H$, the function $\lambda \mapsto \langle E(\lambda)x, y\rangle$ in $\lambda$, is said to be **of bounded variation**; that is, for any division $\Delta$ of $[a, b]$, $\Delta : a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$,

$$\sum_{i=1}^{n} \|(E(\lambda_i) - E(\lambda_{i-1}))x\| \le C < \infty.$$

Here $C$ is a positive constant depending only on $a$, $b$, $x \in H$.

(1.13) Given a resolution of the identity $\{E(\lambda); -\infty < \lambda < \infty\}$ and a real-valued continuous function $f(\lambda)$ on $\mathbf{R}$, and $x \in H$, the integral $\int_a^b f(\lambda)\, dE(\lambda)x$ with values in $H$ on $[a, b]$ is defined as follows: For any division of $[a, b]$, $\Delta : a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$, consider the **Riemann sum**

$$\sum_{i=1}^{n} f(x_i) E(\lambda_{i-1}, \lambda_i] x \in H,$$

where $x_i \in (\lambda_{i-1}, \lambda_i]$ is chosen arbitrarily. Then as $\delta(\Delta) := \max_{1 \le i \le n}(\lambda_i - \lambda_{i-1})$ tends to zero, there exists a limit in $(H, (\ ,\ ))$,

$$\lim_{\delta(\Delta) \to 0} \sum_{i=1}^{n} f(x_i) E(\lambda_{i-1}, \lambda_i] x$$

denoted by $\int_a^b f(\lambda)\, dE(\lambda)x$. We denote by $D_f(H)$, the set of all $x \in H$ such that the Riemann-Stieltjes integral $\int_{-\infty}^{\infty} f(\lambda)\, d\|E(\lambda)x\|^2$ of the function $\lambda \mapsto \|E(\lambda)x\|^2$ of bounded variation, is finite. Then it turns out that $D_f(H)$ is a dense subset of $H$ and for all $x \in D_f(H)$,

$$A_f x := \int_{-\infty}^{\infty} f(\lambda)\, dE(\lambda)x \in H$$

is well defined. In general, for any Borel measurable function $f(\lambda)$, the operator $A_f : D_f(H) \to H$ is well defined. Moreover, if $f$ is bounded measurable, then $D_f(H) = H$ and $A_f$ is well defined on the whole space $H$ and is a bounded selfadjoint linear operator.

(1.14) **Spectral resolution** *of a selfadjoint operator.* Conversely, for any bounded selfadjoint linear operator $A$ with $f(\lambda) = \lambda$, there exists a unique resolution of the identity $\{E(\lambda); -\infty < \lambda < \infty\}$ such that

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda), \quad \text{i.e.,} \; Ax = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) x \in H, \; x \in H.$$

(1.15) *Spectrum of a bounded selfadjoint operator.* For a bounded linear selfadjoint operator $A$ of a Hilbert space $(H, (\ ,\ ))$, the set of all $\lambda$ such that the inverse $(\lambda I - A)^{-1}$ of $\lambda I - A$ is a bounded linear operator of $H$ is called the **resolvent** of $A$, denoted by $\rho(A)$, and its complement is called the **spectrum** of $A$, denoted by $\sigma(A) := \mathbf{R} \backslash \rho(A)$. Then

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) = \int_{\sigma(A)} \lambda \, dE(\lambda),$$

and if we define $r_\sigma(A) := \lim_{n \to \infty} \|A^n\|^{1/n}$, called the **spectral radius** of $A$, then

$$r_\sigma(A) \le \|A\| \quad \text{and} \quad r_\sigma(A) = \sup_{\lambda \in \sigma(A)} |\lambda|. \tag{1.17}$$

### 1.4. Proof of Morse Lemma (1.10).

(1.18) *The first step.* Assume that there exist a neighborhood $V$ of 0 and a $C^k$-diffeomorphism $\psi : V \to V$ with $\psi(0) = 0$ such that $f$ is given by

$$f(v) = (A\psi(v), \psi(v)), \qquad v \in V, \tag{1.19}$$

where $A$ is an invertible selfadjoint operator. Then we prove Lemma (1.10).

Let $A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$ be the spectral resolution of $A$. Let $h(\lambda)$ be the characteristic function of the half interval $[0, \infty)$. That is,

$$h(\lambda) = 0 \quad (-\infty < \lambda < 0) \quad \text{and} \quad h(\lambda) = 1 \quad (0 \le \lambda < \infty).$$

Define

$$P := A_h = \int_{-\infty}^{\infty} h(\lambda) \, dE(\lambda).$$

Then by the definition of $h$,

$$P = \int_0^{\infty} dE(\lambda) = E(\infty) - E(0) = I - E(0).$$

Thus, $P^2 = P$. On the other hand, since $A$ is invertible, $0 \notin \sigma(A)$. Taking $g(\lambda) := |\lambda|^{-1/2}$, we see that $g(\lambda)$ is continuous and never vanishes on $\sigma(A)$. Thus, we can define an invertible selfadjoint operator on $H$ by

$$T := A_g = \int_{-\infty}^{\infty} g(\lambda) \, dE(\lambda)$$

which commutes with $A$. Note that if $\lambda \ne 0$, then

$$\lambda \, g(\lambda)^2 = \lambda \, |\lambda|^{-1} = \operatorname{sgn}(\lambda) = h(\lambda) - (1 - h(\lambda))$$

which implies by (1.13) that

$$AT^2 = P - (I - P). \tag{1.20}$$

Therefore, we can take $\varphi := \psi^{-1} \circ T$ as the $\varphi$ in Lemma (1.10). In fact, $\varphi(0) = 0$, and $\varphi$ is a $C^k$-diffeomorphism of a neighborhood of 0 and satisfies

$$\begin{aligned} f(\varphi(v)) &= f(\psi^{-1}Tv) \\ &= (ATv, Tv) \qquad \text{(by (1.19))} \\ &= (AT^2v, v) \qquad \text{(since } T \text{ is selfadjoint and commutes } A) \\ &= (Pv, v) - ((I-P)v, v) \quad \text{(by (1.20))} \\ &= \|Pv\|^2 - \|(I-P)v\|^2 \end{aligned}$$

since $P$, $I - P$ are projections. This is the desired result.

(1.21) *The second step.* The $\psi$, $A$ in the first step are obtained as follows: For $f$, take $p = 0$, $x = v$, and apply Taylor's theorem ((1.28) and the remark below it in Chapter 2) for $k = m = 2$. Then we get

$$f(x) = f(0) + df_0(v) + R_2(v)(v, v).$$

But since $f(0) = 0$ and $df_0(v) = 0$, we have

$$f(v) = R_2(v)(v, v). \tag{1.22}$$

Then $R_2$ is a bounded symmetric bilinear mapping defined on $U$

$$R_2 : U \ni v \mapsto R_2(v) \in L^2(H; H).$$

Therefore, by a suitable selfadjoint operator $A(v)$ of $H$, $R_2$ can be written as

$$R_2(v)(w, z) = (A(v)w, z), \qquad w, z \in H.$$

Moreover, since $R_2(0) = \frac{1}{2}d^2 f_0$, we obtain

$$\begin{cases} \text{(i)} \ \ f(v) = (A(v)v, v), & v \in U, \\ \text{(ii)} \ \ d^2 f_0(v, w) = 2(A(0)v, w), & v, w \in H, \\ \text{(iii)} \ \ A(0) \text{ is invertible.} \end{cases} \tag{1.23}$$

(iii) follows from the assumption that 0 is a nondegenerate critical point of $f$.

Therefore, define $A := A(0)$, and define $\psi$ as in the following procedure:

By (iii) in (1.23), taking a sufficiently small neighborhood $U$ of 0, we may assume $A(v)$ is invertible for all $v \in U$. So, putting

$$B(v) := A(v)^{-1}A(0), \qquad v \in U,$$

$B$ is a $C^k$-mapping $U \to L(H, H)$, each $B(v)$ is invertible, and $B(0) = I$. Remembering the binomial expansion formula for a real $\alpha$,

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots, \qquad |x| < 1,$$

we define, taking a smaller $U$ if necessary,

$$C(v) := B(v)^{1/2}, \qquad v \in U.$$

Then $C$ is also a $C^k$-mapping $U \to L(H, H)$, and each $C(v)$ is invertible. Then

$$C(v)^* A(v) C(v) = A(0), \quad \text{i.e.,}\ A(v) = C_1(v)^* A(0) C_1(v), \tag{1.24}$$

where $C_1(v) := C(v)^{-1}$. Because, since $A(0)$ and $A(v)$ are selfadjoint,

$$B(v)^* A(v) = (A(v)^{-1} A(0))^* A(v) = A(0) \quad \text{and} \quad A(v)B(v) = A(0).$$

Then

$$B(v)^* A(v) = A(v) B(v).$$

This holds for all polynomials in $B(v)$ and all series in $B(v)$ which are limits of polynomials. Therefore, it holds for $C(v) = B(v)^{1/2}$:

$$C(v)^* A(v) = A(v) C(v).$$

Thus, for the right-hand side of (1.24),

$$C(v)^* A(v) C(v) = A(v) C(v)^2 = A(v) B(v) = A(0)$$

which is (1.24).

So we define

$$\psi(v) := C_1(v) v. \tag{1.25}$$

Then $\psi$ is $C^k$ on the neighborhood $U$ of 0, and furthermore, by (i) of (1.23),

$$\begin{aligned} f(v) &= \langle A(v)v, v\rangle \\ &= \langle C_1(v)^* A(0) C_1(v) v, v\rangle \quad (\text{by } (1.24)) \\ &= \langle A(0) C_1(v) v, C_1(v) v\rangle \quad (\text{by definition of } C_1(v)^*) \\ &= \langle A\psi(v), \psi(v)\rangle \quad (\text{by definition of } A \text{ and } \psi.) \end{aligned}$$

Here we shall show that $d\psi_0 : H \to H$ is an isomorphism. Then by the inverse function theorem ((1.30) of Chapter 2), $\psi$ is a $C^k$-diffeomorphism of a neighborhood of 0 as desired.

(1.26) *The third step.* $d\psi_0 = I : H \to H$. In particular, $d\psi_0$ is an isomorphism.

In fact, differentiating $\psi(v) = C_1(v)v$ in the direction $w$ at $v$, we get

$$\begin{aligned} d\psi_v(w) &= \frac{d}{dt}\Big|_{t=0} C_1(v + tw)(v + tw) \\ &= \frac{d}{dt}\Big|_{t=0} C_1(v + tw)v + C_1(v)w. \end{aligned}$$

Here putting $v = 0$, the first term of the above vanishes, so we get

$$d\psi_0(w) = C_1(0)w, \qquad w \in H.$$

But note that

$$C_1(0) = C(0)^{-1} = B(0)^{-1/2} = I$$

which implies, together with the above, that $d\psi_0 = I$. □

COROLLARY (1.27). *Under the assumption of the Morse Lemma* (1.10), *the index of $f$ at* 0 *coincides with the dimension of*

$$range(I - P) := \{(I - P)u\, ;\, u \in H\}.$$

PROOF. By the equation of the conclusion of the Morse Lemma (1.10), we get the estimate index $\leq$ dim range$(I - P)$. We shall prove the reverse inequality. By definition of the index, it suffices to show that the operator $I - P$ is one to one on $W$ if $W$ is a subspace of $H$ on which $d^2 f_0$ is negative definite. Indeed, this can be shown as follows. Suppose $w \in W$ satisfies $(I - P)w = 0$. Then by the Morse Lemma (1.10),

$$\begin{aligned} d^2 f_0(w, w) &= \frac{d^2}{dt^2}\bigg|_{t=0} \{(P(tw), P(tw)) - \langle (I - P)(tw), (I - P)(tw)\rangle\} \\ &= 2\|Pw\|^2 - 2\|(I - P)w\|^2 = 2\|Pw\|^2 \geq 0. \end{aligned}$$

Therefore, if $w \in W$ is nonzero, then $d^2 f_0(w, w) < 0$, which contradicts the above. Thus, $w = 0$. □

**1.5. The canonical form at a regular point.** By the Morse Lemma (1.10), we determine the form of $f$ at a nondegenarate critical point. In this subsection, we shall show that $f$ is of linear form around a regular point.

PROPOSITION (1.28). *Let $U$ be an open neighborhood of* 0 *in a Banach space $E$, and let $f : U \to \mathbf{R}$ be a $C^k$-function on $U$ satisfying $f(0) = 0$. Assume* 0 *is a regular point of $f$. Then there exists a neighborhood $W$ contained in $U$ on which $f$ is of the following form:*

$$f(\varphi(v)) = \ell(v), \qquad v \in W,$$

*where $\varphi : W \to U$ is an into $C^k$-diffeomorphism and $\ell : E \to \mathbf{R}$ is a bounded linear mapping.*

PROOF. Let $\ell := df_0 \not\equiv 0$. We choose $x \in E$ satifying $\ell(x) = 1$. Let $N := \{v \in E\, ;\, \ell(v) = 0\}$. Then the linear mapping

$$T : E \ni v \mapsto T(v) := (v - \ell(v)x, \ell(v)) \in N \times \mathbf{R}$$

is an isomorphism. The injectivity is clear. The surjectiveness is as follows. For $(v_0, a) \in N \times \mathbf{R}$, we put $v := v_0 + ax$. Then we get $T(v) = (v_0, a)$.

Now define

$$\psi : U \ni v \mapsto \psi(v) := (v - \ell(v)x, f(v)) \in N \times \mathbf{R}.$$

Then $\psi$ is $C^k$ and

$$\begin{aligned} d\psi_u(v) &= \frac{d}{dt}\Big|_{t=0} \psi(u+tv) \\ &= \left( \frac{d}{dt}\Big|_{t=0} (u+tv-\ell(u+tv)x),\ \frac{d}{dt}\Big|_{t=0} f(u+tv) \right) \\ &= (v-\ell(v)x,\ df_u(v)). \end{aligned}$$

In particular, we obtain

$$d\psi_0(v) = (v-\ell(v)x,\ df_0(v)) = T(v), \quad \text{i.e., } d\psi_0 = T.$$

Therefore, by the inverse function theorem ((1.30) in Chapter 2), $\varphi := \psi^{-1}T$ is a $C^k$-diffeomorphism of some neighborhood of 0 in $E$ into $E$ satisfying $\varphi(0) = 0$. Moreover, we have

$$f(\varphi(v)) = \ell(v), \qquad v \in W.$$

Because if we put $v' := \psi^{-1}Tv$, then

$$(v' - \ell(v')x,\ f(v')) = \psi(v') = Tv = (v-\ell(v)x,\ \ell(v)).$$

Therefore, we obtain $f(v') = \ell(v)$. □

COROLLARY (1.29). *Let $M$ be a $C^k$-manifold, $f: M \to \mathbf{R}$ a $C^k$ function, $(k \geq 1)$. Assume that $a \in \mathbf{R}$ be a regular value of $f$. Then the level set $f^{-1}(a)$ is a $C^k$-closed submanifold of $M$.*

PROOF. For $p \in f^{-1}(a)$, let $U_\alpha$, $\alpha: U_\alpha \to E$ be a coordinate neighborhood of $p$. Then define a $C^k$-function $\tilde{f} := f \circ \alpha^{-1} - f(p)$ on $\alpha(U_\alpha) \subset E$. Apply Proposition (1.28) to $\tilde{f}$. There exist a neighborhood $W$ of 0 contained in $\alpha(U_\alpha)$ and a $C^k$-diffeomorphism $\varphi : W \to \alpha(U_\alpha)$ such that $\tilde{f}(\varphi(v)) = \ell(v)$, $v \in W$. Then letting $W_1$, $W_2$ be neighborhoods of 0 in $N = \{v \in E;\ \ell(v) = 0\}$ and $\mathbf{R}$, respectively, satisfying $T^{-1}(W_1 \times W_2) \subset W$, it turns out that

$$U_\alpha := \alpha^{-1}(\varphi(T^{-1}(W_1 \times \{0\})))$$

is the disired coordinate neighborhood of $p$ in $f^{-1}(a)$. □

**1.6. Gradient vector fields.**

(1.30) Let $(M, g)$ be a $C^{k+1}$-Riemannian manifold, and let $f: M \to \mathbf{R}$ be a $C^{k+1}$-function. Then at each point $p \in M$, $df_p : T_pM \to \mathbf{R}$ is a bounded linear mapping, so there exists a unique vector, say $(\nabla f)_p \in T_pM$, satisfying

$$df_p(v) = g_p(v, (\nabla f)_p), \qquad v \in T_pM.$$

We call $(\nabla f)_p$ the **gradient vector** of $f$ at $p$, and

$$\nabla f : M \ni p \mapsto (\nabla f)_p \in T_pM$$

the **gradient vector field** of $f$. (We wrote grad $f$ in (3.25) in Chapter 2. Next (1.31) is clear from (3.26) in Chapter 2 in the case $\dim(M) < \infty$.)

(1.31) The gradient vector field $\nabla f$ is a $C^k$-vector field on $M$.

PROOF. Let $(U_\alpha, \alpha)$, $\alpha : U_\alpha \to H$ be a coordinate neighborhood in $M$, $(H, \langle\ ,\ \rangle)$ a Hilbert space which is a model of $M$. We define a mapping

$$T : H^* \ni \ell \mapsto T\ell \in H$$

by

$$\ell(v) = \langle v, T\ell\rangle, \qquad v \in H,\ \ell \in H^*.$$

Then $T$ is a linear isomorphism, so it is a $C^\infty$-mapping ((1.16) in Chapter 2). The function $h := f \circ \alpha^{-1}$ is a $C^{k+1}$-function on $\alpha(U_\alpha)$, and then the differentiation $dh : U \to H^*$ is a $C^k$-mapping. Thus, $\lambda := T \circ dh : \alpha(U_\alpha) \to H$ is also $C^k$. Moreover, for $x \in U_\alpha$, $v \in H$, we get

$$\begin{aligned}\langle G^\alpha(x) d\alpha_x(\nabla f)_x, v\rangle &= g_x((\nabla f)_x, d(\alpha^{-1})v) \quad \text{(by (2.32) in Chapter 2)}\\ &= df_x(d(\alpha^{-1})v) \quad \text{(by definition of } \nabla f)\\ &= dh_{\alpha(x)}(v) \quad \text{(definition of } h \text{ and the composition rule)}\\ &= \langle v, T\,dh_{\alpha(x)}\rangle \quad \text{(by definition of } T).\end{aligned}$$

Since $v \in H$ is arbitrary, we get

$$d\alpha_x(\nabla f)_x = G^\alpha(x)^{-1}\, T\,dh_{\alpha(x)},$$

where $x \mapsto G^\alpha(x)^{-1} \in L(H, H)$ is $C^k$, and $x \mapsto T\,dh_{\alpha(x)}$ is $C^k$, so we can conclude that $x \mapsto d\alpha_x(\nabla f)_x$ is $C^k$ as desired. □

The following is clear:

(1.32) $(\nabla f)_p = 0 \iff p$ is a critical point of $f$.

Thus, the set of all critical points coincides with the nullset of $\|\nabla f\|$. Moreover, since

$$(\nabla f)f(p) = df_p((\nabla f)_p) = g_p((\nabla f)_p, (\nabla f)_p) = \|(\nabla f)_p\|^2,$$

$(\nabla f)f > 0$ on the set of regular points of $f$.

## §2. Minimum values of smooth functions

In this section, we show which smooth functions admit minima. In the following argument, the condition (C) given by Palais and Smale is essential.

**2.1. The condition (C).** Assume that $(M, g)$ is a $C^{k+1}$-Riemannian manifold and that $f : M \to \mathbf{R}$ is a $C^{k+1}$-function ($k \geq 1$).

DEFINITION (2.1). $f$ satisfies the **condition (C)** if the following holds: Assume a subset $S$ of $M$ satisfies (2.2).

$$f \text{ is bounded on } S, \text{ and } \inf_S \|\nabla f\| = 0. \tag{2.2}$$

Then there exists a point $p$ in the closure $\overline{S}$ of $S$ such that $p$ is a critical point of $f$, i.e., $\nabla f_p = 0$.

EXAMPLE. Consider the following two functions on $M = \mathbf{R}$ given of §1 in Chapter 1:

(i) $f(x) = x^2, \quad -\infty < x < \infty,$

(ii) $g(x) = e^{x^3}, \quad -\infty < x < \infty.$

Then $f$ satisfies the condition (C) and takes a minimum at $x = 0$, but $g$ does not satisfy the condition (C) and takes no minimum.

In fact, if we take $S := (-\infty, a]$, then $g(x) > 0$ on $S$ and it follows that $\inf_{x \in S} g'(x) = 0$. But there is no element $x$ in $S = \overline{S}$ such that $g'(x) = 0$; thus, $g$ does not satisfy the condition (C).

By Weierstrass' Theorem, any continuous function on a compact set admits both minimum and maximum. But this is a rather delicate problem for a continuous function on a noncompact set. The above simple example suggests that the condition (C) is essential. The condition (C) is very similar condition to "a given function is defined on a compact set ". It seems that the naming of the condition (C) by Palais and Smale comes from "compactness". We shall show that under the condition (C), every thing is OK as on a compact set.

### 2.2. Minima of smooth functions.

PROPOSITION (2.3). *Let $(M, g)$ be $C^{k+2}$-Riemannian manifold, and let $f : M \to \mathbf{R}$ be a $C^{k+2}$-function $(k \geq 1)$. Assume that $f$ satisfies the condition* (C) *and admits only nondegenerate critical points. Then for any two real numbers $a < b$, the set of all critical points $p$ of $f$ satsifying $a < f(p) < b$ is finite. In particular, if $c$ is a critical value of $f$, then the set of all critical points of $f$ contained in $f^{-1}(c)$ is finite.*

PROOF. Assume that the conclusion is false. Then there exists a sequence of mutually distinct critical points $\{p_n\}_{n=1}^{\infty}$ of $f$ satisfying $a < f(p_n) < b$. Since $p_n$ is nondegenerate, by the Morse Lemma (1.10), there exists a neighborhood such that $f$ admits no other critical point. Thus, such critical points are isolated. Thus, for each $n$, there exists a regular point $q_n$ of $f$ satisfying that

$$\rho(p_n, q_n) < \frac{1}{n}, \qquad \|\nabla f_{q_n}\| < \frac{1}{n} \quad \text{and} \quad a < f(q_n) < b.$$

Then we apply the condition (C) to the set $S := \{q_n ; n = 1, 2, \ldots\}$ and choose a subsequence $\{q_{n_k}\}_{k=1}^{\infty}$ of $\{q_n\}_{n=1}^{\infty}$ convergent to some point $p$ which is a critical point of $f$. But, by $\rho(p_{n_k}, q_{n_k}) < 1/n_k$, $p_{n_k}$ converge to $p$. This implies that nondegenerate critical points of $f$ converge to a critical point of $f$, which is a contradiction. □

PROPOSITION (2.4). *Let $(M, g)$ be a complete $C^{k+1}$-Riemannian manifold, and let $f : M \to \mathbf{R}$ be a $C^{k+1}$-function satisfying the condition* (C),

$(k \geq 1)$. *Let* $\sigma : (\alpha, \beta) \to M$ *be a maximal integral curve of* $\nabla f$ *defined an open interval* $(\alpha, \beta)$. *Then we have*

(i) *either* $\lim_{t\to\beta} f(\sigma(t)) = \infty$, *or* $\beta = \infty$ *and if* $t \to \infty$, *then* $\sigma(t)$ *admits a critical point as accumulation point.*

(ii) *Either* $\lim_{t\to\alpha} f(\sigma(t)) = -\infty$, *or* $\alpha = -\infty$ *and if* $t \to -\infty$, *then* $\sigma(t)$ *admits a critical point as accumulation point.*

Before going into the proof, we prepare with the following:

We define the **length** of a $C^1$-curve defined on an open interval. For a $C^1$-curve $\sigma : (a, b) \to M$, the length of $\sigma$ is by definition

$$L(\sigma) := \lim_{\substack{\alpha\to a\\ \beta\to b}} \int_\alpha^\beta \|\sigma'(t)\|\, dt.$$

$L(\sigma)$ may be infinite. If $L(\sigma) < \infty$, then for any $\epsilon > 0$, there exists a division of $(a, b)$, $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, such that, for any $i = 1, \cdots, n$,

$$\int_{t_{i-1}}^{t_i} \|\sigma'(t)\|\, dt < \epsilon.$$

Therefore, if $B_r(p) := \{x \in M;\ \rho(x, p) < r\}$ for $r > 0$ and $p \in M$,

$$\sigma((a, b)) \subset \bigcup_{i=1}^n B_\epsilon(\sigma(t_i)).$$

That is, $\sigma((a, b))$ is totally bounded. Therefore, we get

LEMMA (2.5). *Let* $(M, g)$ *be a* $C^{k+1}$-*Riemannian manifold,* $(k \geq 1)$, *and let* $\sigma : (a, b) \to M$ *be a* $C^1$-*curve with* $L(\sigma) < \infty$. *Then* $\sigma((a, b))$ *is totally bounded in* $M$. *In particular, if* $(M, g)$ *is complete, then the closure of* $\sigma((a, b))$ *is compact.*

LEMMA (2.6). *Let* $(M, g)$ *be a complete* $C^{k+1}$-*Riemannian manifold* $(k \geq 1)$, *let* $X$ *be a* $C^k$-*vector field on* $M$, *and let* $\sigma : (a, b) \to M$ *be a maximal integral curve of* $X$. *Then*

(i) *if* $b < \infty$, $\int_0^b \|X(\sigma(t))\|\, dt = \infty$. *In particular,* $\|X(\sigma(t))\|$ *is unbounded on* $[0, b)$.

(ii) *If* $-\infty < a$, *then* $\int_a^0 \|X(\sigma(t))\|\, dt = \infty$. *In particular,* $\|X(\sigma(t))\|$ *is unbounded on* $(a, 0]$.

PROOF. Since $\sigma(t)$ is an integral curve of $X$, $\sigma'(t) = X(\sigma(t))$. Therefore, if $\int_0^b \|X(\sigma(t))\|\, dt < \infty$, we get $\int_0^b \|\sigma'(t)\|\, dt < \infty$. By Lemma (2.5), the closure of $\sigma((0, b))$ is compact. Then $\sigma(t)$ has an accumulation point if $t \to b$. This contradicts Theorem (2.29) in Chapter 2. (ii) follows by a similar argument. □

PROOF OF PROPOSITION (2.4). We prove (i). (ii) follows by a similar argument. Let $g(t) := f(\sigma(t))$. Then

$$g'(t) = df_{\sigma(t)}(\sigma'(t)) = df_{\sigma(t)}(\nabla f_{\sigma(t)}) = \|\nabla f_{\sigma(t)}\|^2 \geq 0.$$

Thus, $g(t)$ is nondecreasing, so $\lim_{t\to\beta} g(t)$ exists, say $B$. Assume that $B < \infty$. We shall show $\beta = \infty$ and $\sigma(t)$ has an accumulation point which is a critical point of $f$ as $t \to \infty$. Note that

$$g(t) = g(0) + \int_0^t g'(s)\,ds = g(0) + \int_0^t \|\nabla f_{\sigma(t)}\|^2\,ds.$$

Thus, by the assumption that $B < \infty$, we get $\int_0^\beta \|\nabla f_{\sigma(t)}\|^2\,ds < \infty$. By the Schwarz inequality,

$$\int_0^\beta \|\nabla f_{\sigma(s)}\|\,ds \le \beta^{1/2}\left(\int_0^\beta \|\nabla f_{\sigma(t)}\|^2\,ds\right)^{1/2}.$$

Therefore, if $\beta < \infty$, the right-hand side of the above is finite which implies $\int_0^\beta \|\nabla f_{\sigma(s)}\|\,ds < \infty$. But this contradicts (i) of Lemma (2.6). Thus, $\beta = \infty$.

Moreover, since we get $\int_0^\infty \|\nabla f_{\sigma(s)}\|^2\,ds < \infty$, we have

$$\liminf_{s\to\infty} \|\nabla f_{\sigma(s)}\|^2 = 0.$$

Thus, there exists a sequence $\{s_n\}_{n=1}^\infty$ which converges to $\infty$ and $\|\nabla f_{\sigma(s_n)}\| \to 0$. The assumption that $B < \infty$ implies that $\{f(\sigma(s))\,;\, 0 \le s < \infty\}$ is bounded, so applying the condition (C) to

$$S := \{\sigma(s_n)\,;\, n = 1, 2, \dots\},$$

$\sigma(t)$ has an accumulation point which is a critical point of $f$ as $t \to \infty$.
□

**Proposition (2.7).** *Let $M$ be a connected $C^1$-manifold let $f : M \to \mathbf{R}$ be a nonconstant $C^1$-function, and let $K$ be the set of all critical points of $f$. Then we have*

$$f(K) = f(\partial K),$$

*where $\partial K := \overline{K} - K^\circ$ is the topological boundary of $K$ ($K^\circ$ is the interior of $K$).*

**Proof.** It suffices to show that for all $p \in K$, there exists an $q \in \partial K$ such that $f(x) = f(p)$. See Figure 3.3, on next page. Since $f$ is nonconstant, there exists $q \in M$ such that $f(q) \neq f(p)$. Connect $p$ and $q$ by a $C^1$-curve:

$$\sigma : I = [0, 1] \to M, \qquad C^1, \qquad \sigma(0) = p,\ \sigma(1) = q.$$

Define $g(t) := f(\sigma(t))$. Then $g'(t) = df_{\sigma(t)}(\sigma'(t))$ and $g$ is nonconstant, so $g' \not\equiv 0$. Thus, $\sigma(I) \not\subset K$. Let

$$t_0 := \inf\{t \in I\,;\ \sigma(t) \notin K\},$$

and let $x := \sigma(t_0)$. Then $x \in \partial K$ and $\sigma'(t) = 0$ for all $t \in [0, t_0]$. Thus, $g(t_0) = g(0)$. Therefore,

$$f(x) = f(\sigma(t_0)) = g(t_0) = g(0) = f(p). \quad \square$$

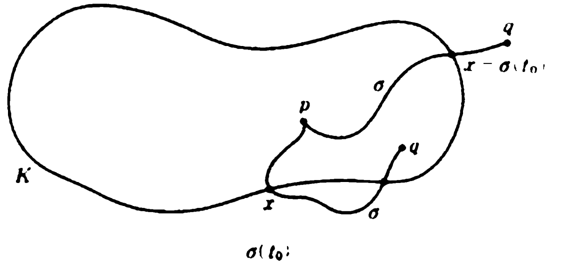



FIGURE 3.3

PROPOSITION (2.8). *Let* $(M, g)$ *be a* $C^1$*-Riemannian manifold, let* $f: M \to \mathbf{R}$ *a* $C^1$*-function satisfying the condition* (C), *and let* $K$ *be the set of all critical points of* $f$. *Then for any two real numbers* $a < b$, $\partial K \cap f^{-1}([a, b])$ *is compact.*

PROOF. Let a sequence $\{p_n\}_{n=1}^{\infty}$ in the set $\partial K$ satisfying $a \leq f(p_n) \leq b$. We have to show that there is a convergent subsequence.

Since $p_n \in \partial K$, we may choose the regular points $q_n$ of $f$ which are arbitrarily close to $p_n$. Since $\|\nabla f\|$ is continuous and $\|\nabla f_{p_n}\| = 0$, we can choose the $q_n$ in such a way that

$$\|\nabla f_{q_n}\| < \frac{1}{n}, \qquad a - \epsilon < f(q_n) < b + \epsilon, \qquad \text{and} \qquad \rho(p_n, q_n) < \frac{1}{n}.$$

We apply the condition (C) to $S := \{q_n;\ n = 1, 2, \ldots\}$, so there exists a subsequence $\{q_{n_k}\}_{k=1}^{\infty}$ of $\{q_n\}_{n=1}^{\infty}$ which is convergent to some point of $K$, say $p$. Then by $\rho(p_{n_k}, q_{n_k}) < 1/n_k$, $p_{n_k}$ converges to $p$. □

THEOREM (2.9). *Let* $(M, g)$ *be a complete* $C^2$*-Riemannian manifold, and let* $f: M \to \mathbf{R}$ *be a* $C^2$*-function satisfying the condition* (C). *Assume that* $f$ *is bounded below on a connected component* $M_0$ *of* $M$. *Then the restriction of* $f$ *to* $M_0$ *attains this infimum.*

PROOF. We need prove the theorem only in the case $M = M_0$. Let $B := \inf\{f(x);\ x \in M\}$. By definition, for any $\epsilon > 0$, there exists $p \in M$ such that $f(p) < B + \epsilon$.

Let $\sigma: (\alpha, \beta) \to M$ be a maximal integral curve of $\nabla f$. By the assumption that $f$ is bounded below and by (ii) of Proposition (2.4), $\alpha = -\infty$ and $\sigma(t)$ has an accumulation point which is a critical point of $f$ as $t \to -\infty$. By the proof of Proposition (2.4), $f(\sigma(t))$ is a nondecreasing function in $t$, so

$$f(q) \leq f(p) < B + \epsilon.$$

Since our theorem holds trivially in the case that $f$ is constant, we may assume $f$ is not constant. So by Proposition (2.7), $f(K) = f(\partial K)$. Then there exists $x \in \partial K$ such that

$$f(x) = f(q) < B + \epsilon.$$

Thus, for $\epsilon = \frac{1}{n}$, we may take $x_n \in \partial K$ such that

$$B \le f(q) = f(x_n) < B + \frac{1}{n}.$$

However, by Proposition (2.8), $f^{-1}([B, B+1]) \cap \partial K$ is compact. Thus, $\{x_n\}_{n=1}^{\infty}$ has a covergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$ convergent to some point $z$. Then

$$B \le f(x_{n_k}) < B + \frac{1}{n_k}$$

which implies that $f(z) = B$. □

Theorem (2.9) is what we want to show. The next theorem is also important for determining the topology of $M$, but we do not need it in our subsequent arguments.

STRONG TRANSVERSALITY THEOREM (2.10). *Let $(M, g)$ be a complete $C^{k+1}$-Riemannian manifold and let $f: M \to \mathbf{R}$ be a $C^{k+1}$-function satisfying the condition* (C), $(k \ge 1)$. *Assume that $f$ admits no critical value in the interval $[a, b]$. Then $M_a := \{x \in M;\ f(x) \le a\}$ is $C^{k+1}$-diffeomorphic to $M_b := \{x \in M;\ f(x) \le b\}$.*

To outline the proof we first consider an integral curve of $\nabla f$ starting any point of $M_a$. Then assigning the first point of the integral curve crossing $M_b$ we get a mapping from $M_a$ to $M_b$. By using the assumption that $f$ has no critical value in $[a, b]$, it can be shown that this mapping is a $C^{k+1}$-diffeomorphism of $M_a$ onto $M_b$.

**2.3. Finsler manifolds.** We should extend Morse theory from the case of Riemannian manifolds to the case of Banach manifolds in order to apply it to $L_{1,p}(M, N)$, $1 > \frac{m}{p}$, where $m = \dim(M)$.

(2.11) *Finsler metric.* A $C^k$-manifold $M$ treated in subsection **2.1** in Chapter 2 was a manifold modelled on a Banach space. In general, it is not true that it admits a Riemannian metric $g$. However, we can consider a Finsler metric $\|\ \|$ instead.

DEFINITION. $\|\ \|$ is called a **Finsler metric** if for each point $x \in M$, $\|\ \|_x$ is a norm on a Banach space $T_xM$ such that:

(i) The topology induced from $\|\ \|_x$ coincides with the original one as its Banach space.

(ii) (Local triviality) If we specify the local triviality of the tangent bundle $T(M)$ for each point $p \in M$ using a coordinate neighborhood $(U_\alpha, \alpha)$, $\alpha: U_\alpha \to E$,

$$\psi_\alpha : U_\alpha \times E \to \pi^{-1}(U_\alpha) \subset T(M),$$

where $\pi;\ T(M) \to M$ is the projection, and define the norm on $E$ through $\psi_\alpha$ by

$$|||v|||_x := \|\psi_\alpha(x, v)\|_x, \qquad x \in U_\alpha,\ v \in E$$

then there exists a constant $C > 0$ such that

$$\frac{1}{C}|||v|||_p \le |||v|||_x \le C\,|||v|||_p, \qquad v \in E,\ x \in U_\alpha.$$

A Banach manifold with such a Finsler metric $\|\ \|$ is called a **Finsler manifold.** In this case, we can define the **length** $L(\sigma)$ of a $C^1$-curve $\sigma$ : $[a, b] \to M$ by

$$L(\sigma) := \int_a^b \|\sigma'(t)\|\,dt.$$

If $p$, $q \in M$ belong to the same connected component of $M$, then the **distance** $\rho(p, q)$ is defined by

$$\rho(p, q) := \inf\{L(\sigma);\ \sigma \text{ is a } C^1\text{-curve connecting } p,\ q\}.$$

Indeed, it turns out that $\rho$ satisfies the three axioms of distance, and the topology on $M$ induced by $\rho$ coincides with the original one on $M$. When the metric space $(M, \rho)$ is complete, a Finsler manifold $M$ is called **complete.** We simply write $M$ to abbreviate a Finsler metric $\|\ \|$.

(2.12) A Finsler metric on the cotangent bundle of a Finsler manifold $M$ is defined as follows: for $p \in M$ and $\ell \in T_p^*M$,

$$\|\ell\| := \sup\{|\ell(u)|;\ u \in T_pM,\ \|u\| = 1\}.$$

Then it is clear that if $f\colon M \to \mathbf{R}$ is a $C^1$-function, then $M \ni p \mapsto \|df_p\|$ is a continuous function on $M$.

(2.13) *The condition* (C). A $C^k$-function $f : M \to \mathbf{R}$ on a Finsler $C^k$-manifold $M$ satisfies the **condition (C)** if a subset $S$ of $M$ has the property that if

$$f \text{ is bounded on } S \quad \text{and} \quad \inf_S \|df\| = 0,$$

then there exists a point $p$ in the closure $\overline{S}$ of $S$ such that $df_p = 0$; that is, $p$ is a critical point of $f$.

Then we use the following **pseudogradient vector field** $X$ instead of the gradient vector field $\nabla f$ of $f$ in the argument in subsection **2.2.**

DEFINITION (2.14). Let $M$ be a Finsler $C^{k+1}$-manifold and let $f : M \to \mathbf{R}$ be a $C^{k+1}$-function ($k \ge 0$). Then a tangent vector $X \in T_pM$ at a point $p$ is called a **pseudogradient vector** of $f$ at $p$ if

(i) $\|X\| \le 2\|df_p\|$,

(ii) $Xf = df_p(X) \ge \|df_p\|^2$.

Moreover, if at each point $p$ of an open subset $S$ of $M$, $X_p$ is a pseudogradient vector of $f$ at $p$ and $X$ is $C^k$ on $S$, then $X$ is called a **$C^k$-pseudogradient vector field** on $S$.

The existence of such a pseudo gradient vector can be shown as follows:

If $p$ is a critical point of $f$, i.e., $df_p = 0$, then the zero vector of $T_pM$ is the pseudogradient vector of $f$ at $p$. If $p$ is a regular point of $f$, then we first choose $Y \in T_pM$ for any $0 < \epsilon < 1$, in such a way that

$$\|Y\| = 1 \quad \text{and} \quad df_p(Y) \geq (1-\epsilon)\,\|df_p\|.$$

This follows from the definition of $\|df_p\|$ in (2.12). So for any $\delta > 0$, letting

$$X := \frac{1+\delta}{1-\epsilon}\,\|df_p\|\,Y,$$

then

$$\|X\| = \frac{1+\delta}{1-\epsilon}\,\|df_p\|$$

and

$$Xf = df_p(X) = \frac{1+\delta}{1-\epsilon}\,\|df_p\|\,df_p(Y) \geq (1+\delta)\,\|df_p\|^2.$$

Therefore, we can choose $\delta > 0$ and $0 < \epsilon < 1$ as $\|X\|^{\cdot}$ is arbitrarily close to $\|df_p\|$ such that $Xf > \|df_p\|^2$. (Note that the number '2' in (i) of Definition (2.14) can be replaced to any number bigger than 1. )

LEMMA (2.15). *Let $M$ be a Finsler $C^{k+1}$-manifold and let $f : M \to \mathbf{R}$ be a $C^{k+1}$-function ($k \geq 0$). If $p \in M$ is a regular point of $f$, then we can choose an open neighborhood $U$ of $p$ on which there exists a $C^k$-pseudogradient vector field.*

PROOF. For $p \in M$, we choose $X_p \in T_pM$ such that

$$\|X_p\| < 2\,\|df_p\| \quad \text{and} \quad X_pf > \|df_p\|^2.$$

This can be shown by the above argument. Here we extend $X_p$ to a $C^k$-vector field on a neighborhood $V$ of $p$. Then letting

$$U := \{q \in V\,;\ X_qf > \|df_q\|^2 \quad \text{and} \quad \|X_q\| < 2\,\|df_q\|\},$$

$U$ is an open set containing $p$ and the desired result follows because $Xf$, $\|df\|$, $\|X\|$ are continuous on $V$. □

PROPOSITION (2.16). *Let $M$ be a Finsler $C^2$-manifold, let $f : M \to \mathbf{R}$ be a $C^1$-function, and let $M^*$, be an open submanifold of $M$ consisting of regular points of $f$. Then there exists a pseudogradient vector field on $M^*$ which is locally Lipschitz continuous.*

Here a vector field $X^*$ is locally **Lipschitz continuous** on $M^*$ if for each $p \in M^*$, and each coordinate neighborhood $(U_\alpha, \alpha)$, $\alpha : U_\alpha \to E$, with the local triviality of $T(M)$ given by $\psi_\alpha : U_\alpha \times E \to \pi^{-1}(U_\alpha) \subset T(M)$, where

$$\psi_\alpha^{-1}(X_x) = (x, X_\alpha(x)), \qquad x \in U_\alpha,\ X_\alpha(x) \in E,$$

then there exists a positive constant $C$ such that

$$\|\|X_\alpha(x) - X_\alpha(y)\|\|_p \leq C\,\|\alpha(x) - \alpha(y)\|, \qquad x, y \in U_\alpha.$$

Outline of the proof of Proposition (2.16). For each $p \in M^*$, let $U_p$ be a neighborhood of $p$ in $M^*$ and let $X^{(p)}$, be a $C^1$-pseudovector field on $U_p$. $M^*$ is paracompact (see definition in subsection 3.6 of Chapter 2), because $M^*$ admits a distance induced from the distance $\rho$ of $M$ and any metric space is always paracompact. Therefore, there exists an open covering $\{U_\beta; \beta \in B\}$ which is a refinement of an open covering $\{U_p; p \in M^*\}$ and is locally finite. Then there exists a partition of unity $\{\varphi_\beta\}_{\beta \in B}$ (in the sense of Lipschitz) corresponding to $\{U_\beta; \beta \in B\}$. That is,

(i) $0 \le \varphi_\beta(x) \le 1$, $x \in M^*$.

(ii) $\mathrm{supp}(\varphi_\beta) \subset U_\beta$ for each $\beta \in B$.

(iii) $\sum_{\beta \in B} \varphi_\beta(x) = 1$ for each $x \in M^*$.

For each $p \in U_\beta$, there exist an open neighborhood $V$ of $p$ contained in $U_\beta$ and a positive constant $C$ such that

$$|\varphi_\beta(x) - \varphi_\beta(y)| \le C \, \|\beta(x) - \beta(y)\|, \quad \text{for all } x, y \in V.$$

Then for each $\beta \in B$, let $U_\beta$ be an open neighborhood of $p(\beta)$ in Lemma (2.15), and let $X^{p(\beta)}$ be a $C^1$-pseudogradient vector field on $U_\beta$. Define

$$X := \sum_{\beta \in B} \varphi_\beta X^{p(\beta)},$$

which is the desired pseudo gradient vector field of $f$ on $M^*$. □

Using the pseudogradient vector field $X$ instead of $\nabla f$, the following theorem can be shown.

Theorem (2.17). *Let $M$ be a complete Finsler $C^2$-manifold, and let $f : M \to \mathbf{R}$ be a $C^2$-function satisfying the condition* (C). *Then*

(i) *if $f$ is bounded below on a connected conponent $M_0$ of $M$, then $f$ attains a minimum on $M_0$.*

(ii) *For any two reals $a < b$, if $f$ has no critical value on $[a, b]$, then $M_a := \{x \in M; f(x) \le a\}$ is locally Lipschitz diffeomorphic to $M_b := \{x \in M; f(x) \le b\}$.*

(This can be shown by a similar argument to the one in §2 and so is omitted. Refer to R.S. Palais [P2] for a proof. )

## §3. The condition (C)

**3.1. Main Theorem.** In §2, we showed that any $C^2$-function with the condition (C) which is bounded below on a connected component of a manifold attains a minimum there. So our next problem is to clarify which functions satisfy the condition (C). But a general theory is still unknown. We only show Theorem (3.3) about $L_{1,p}(M, N)$ following [P3].

Let us recall the situation in subsection 4.3 in Chapter 2. Let $(M, g)$, $(N, h)$ be compact $m$-, $n$-dimensional $C^\infty$-Riemannian manifolds, respectively. We assume $(N, h)$ satisfies

(i) $N$ is a closed submanifold of $\mathbf{R}^K$, and it holds that
(ii) $h = \iota^* g_0$, where $\iota : N \subset \mathbf{R}^K$ is the inclusion and $g_0$ is the standard Riemannian metric on $\mathbf{R}^K$.

In this setting, we consider the Banach space $L_{1,p}(M, \mathbf{R}^K)$ and the $C^\infty$-manifold $L_{1,p}(M, N)$ ( $1 > \frac{m}{p}$, where $m = \dim(M)$ ).

(3.1) *Finsler metric.* We define a Finsler metric $\| \ \|$ on the Banach space $L_{1,p}(M, \mathbf{R}^K)$ as follows: For any element $v$ of the tangent space $T_u L_{1,p}(M, \mathbf{R}^K) \cong L_{1,p}(M, \mathbf{R}^K)$ of $L_{1,p}(M, \mathbf{R}^K)$ at $u$, define

$$\|v\|_{1,p} := \left( \int_M |dv|^p v_g + \int_M |v|^p v_g \right)^{1/p}.$$

Here for $v \in L_{1,p}(M, \mathbf{R}^K)$, we denote $v = (v_1, \cdots, v_K)$, and

$$|dv|_x := \left( \sum_{A=1}^{K} |dv_A|_x{}^2 \right)^{1/2}, \quad x \in M,$$

where $|d_A|_x$ is the norm of 1-form $dv_A$ with respect to the inner product on the cotangent space $T_x^* M$ induced from the Riemannian metric $g$. And $|v|$ is given by

$$|v|_x := \left( \sum_{A=1}^{K} |v_A|_x{}^2 \right)^{1/2}, \quad x \in M.$$

Then $\| \ \|_{1,p}$ defines a Finsler metric on the Banach space $L_{1,p}(M, \mathbf{R}^K)$, and we denote the induced distance by $\rho_0$ which is given by

$$\rho_0(u, v) = \|u - v\|_{1,p}, \qquad u, v \in L_{1,p}(M, \mathbf{R}^K).$$

Since the metric space $(L_{1,p}(M, \mathbf{R}^K), \rho_0)$ is complete, the Finsler metric $\| \ \|_{1,p}$ on $L_{1,p}(M, \mathbf{R}^K)$ is complete (cf. (2.11)).

Next since $L_{1,p}(M, N)$ is a closed submanifold of the Banach space $L_{1,p}(M, \mathbf{R}^K)$ (cf. Theorem (4.33) in Chapter 2), we can give a Finsler metric, denoted by the same letter $\| \ \|_{1,p}$, on $L_{1,p}(M, N)$ from the pull back of $\| \ \|_{1,p}$ on $L_{1,p}(M, \mathbf{R}^K)$ by the inclusion $\iota : N \subset \mathbf{R}^K$. Then for each $\phi \in L_{1,p}(M, N)$ and for

$$X \in T_\phi L_{1,p}(M, N) = L_{1,p}(\phi^{-1} TN) \subset L_{1,p}(M, \mathbf{R}^K),$$

we get

$$\|X\|_{1,p} = \left( \int_M |dX|^p v_g + \int_M |X|^p v_g \right)^{1/p}.$$

Now denote by $\rho$ the distance with respect to the Finsler metric $\| \ \|_{1,p}$ on $L_{1,p}(M, N)$. Then this distance has the property that for $\phi_1, \phi_2 \in L_{1,p}(M, N)$,

$$\rho(\phi_1, \phi_2) := \inf\{L(\gamma); \gamma \text{ is a } C^1\text{-curve in } L_{1,p}(M, N) \text{ connecting } \phi_1, \phi_2\}$$
$$\geq \inf\{L(\gamma); \gamma \text{ is a } C^1\text{-curve in } L_{1,p}(M, \mathbf{R}^K) \text{ connecting } \phi_1, \phi_2\}$$
$$=: \rho_0(\phi_1, \phi_2).$$

Moreover, we get

PROPOSITION (3.2). *The Finsler manifold* $(L_{1,p}(M, N), \| \|_{1,p})$ *is complete.*

PROOF. Let $\{\phi_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $L_{1,p}(M, N)$ with respect to $\rho$. Then by the above inequality, it is also a Cauchy sequence in $L_{1,p}(M, \mathbf{R}^K)$ with respect to $\rho_0$. But $(L_{1,p}(M, \mathbf{R}^K), \rho_0)$ is complete, so this sequence is convergent:

$$\phi_i \to \phi_0 \in L_{1,p}(M, \mathbf{R}^K) \quad \text{as } i \to \infty.$$

However, $L_{1,p}(M, N)$ is closed in $L_{1,p}(M, \mathbf{R}^K)$, so $\phi_0 \in L_{1,p}(M, N)$. □

Next we define the function $\mathcal{J}$ on $L_{1,p}(M, \mathbf{R}^K)$ by

$$\mathcal{J}(u) := \int_M |du|^p v_g, \qquad u \in L_{1,p}(M, \mathbf{R}^K).$$

Moreover, we define the function $J$ on $L_{1,p}(M, N)$ to be the restriction of $\mathcal{J}$:

$$J(\phi) := \mathcal{J}(\phi), \qquad \phi \in L_{1,p}(M, N).$$

Then our main theorem is as follows

MAIN THEOREM (3.3). *Let* $(M, g)$, $(N, h)$ *be compact* $m$-, $n$-*dimensional Riemannian manifolds, respectively. Assume that* $N$ *is a closed submanifold of* $\mathbf{R}^K$ *and* $h = \iota^* g_0$, *where* $g_0$ *is the standard Riemannian metric on* $\mathbf{R}^K$. *Assume that* $1 > \frac{m}{p}$, *where* $m = \dim(M)$. *Then the above function* $J : L_{1,p}(M, N) \to \mathbf{R}$ *satisfies*

(i) $J$ *is a* $C^2$-*function on* $L_{1,p}(M, N)$,

*and*

(ii) $J$ *satisfies the condition* (C).

(3.4) PROOF OF (i) OF THEOREM (3.3). Since $L_{1,p}(M, N)$ is a closed $C^\infty$-manifold of $L_{1,p}(M, \mathbf{R}^K)$, it suffices to show that $\mathcal{J}$ is $C^2$ on $L_{1,p}(M, \mathbf{R}^K)$.

For $u, v, w \in L_{1,p}(M, \mathbf{R}^K)$, we have

$$d\mathcal{J}_u(v) = p \int_M \langle du, du\rangle^{p/2-1} \langle du, dv\rangle v_g. \tag{3.5}$$

Indeed, we may calculate

$$d\mathcal{J}_u(v) = \frac{d}{dt}\bigg|\mathcal{J}(u+tv) = \frac{d}{dt}\bigg|\int_M (d(u+tv), d(u+tv))^{p/2} v_g.$$

Now we use Hölder's inequality: letting $r' = r/(r-1)$,

$$\int_M |f_1 f_2| v_g \le \|f_1\|_r \|f_2\|_{r'}, \qquad f_1 \in L_r(M), \; f_2 \in L_{r'}(M). \tag{3.6}$$

Since $|(du, dv)| \le |du|\,|dv|$, we get

$$\begin{aligned}
|\text{right-hand side of (3.5)}| &\le p\int_M |du|^{p-2}|du|\,|dv|\, v_g \\
&= p\int_M |du|^{p-1}|dv|\, v_g \\
&\le p\left(\int_M |du|^p v_g\right)^{(p-1)/p}\left(\int_M |dv|^p v_g\right)^{1/p} \\
&\le p\,(\|u\|_{1,p})^{p-1}\|v\|_{1,p},
\end{aligned}$$

where we use (3.6) in the last step putting $r = p/(p-1)$, $r' = p$. Therefore, $d\mathcal{J}_u : L_{1,p}(M, \mathbf{R}^K) \ni v \mapsto d\mathcal{J}_u(v) \in \mathbf{R}$ is a bounded linear mapping.

Furthermore, for $d^2\mathcal{J}_u$,

$$\begin{aligned}
d^2\mathcal{J}_u(v, w) &= \frac{\partial^2}{\partial t\partial s}\bigg|_{(t,s)=(0,0)}\int_M |d(u+tv+sw)|^p v_g \\
&= \frac{d}{dt}\bigg|_{t=0} p\int_M (d(u+tv), d(u+tv))^{p/2-1}(d(u+tv), dw)\, v_g \\
&= p(p-2)\int_M (du, du)^{p/2-2}(du, dv)(du, dw)\, v_g \\
&\quad + p\int_M (du, dv)^{p/2-1}(dv, dw)\, v_g.
\end{aligned}$$

Using Hölder's inequality in a similar way, it turns out $d^2\mathcal{J}_u : L_{1,p}(M, \mathbf{R}) \times L_{1,p}(M, \mathbf{R}) \to \mathbf{R}$ is a bounded bilinear mapping, and due to Zorn's Proposition (1.12) in Chapter 2, $\mathcal{J}$ is $C^2$.

**3.2. Proof of the condition (C).** We shall prove the following two lemmas in subsections 3.3 and 3.4. Here we prove (ii) of Main Theorem (3.3) under these lemmas.

LEMMA (3.7). *Assume that a sequence $\{\phi_i\}_{i=1}^{\infty}$ in $L_{1,p}(M, N)$ is bounded in $L_{1,p}(M, \mathbf{R}^K)$ that $\|dJ_{\phi_i}\| \to 0$ as $i \to \infty$. Then there exists a subsequence, denoted by the same letter for simplicity, such that*

$$d\mathcal{J}_{\phi_i}(\phi_i - \phi_j) \to 0 \quad \text{as } i, j \to \infty.$$

REMARK. Remember that for a $C^2$-function $J : L_{1,p}(M, N) \to \mathbf{R}$ and $\phi \in L_{1,p}(M, N)$, the differentiation of $J$ at $\phi$, $dJ_\phi$ is a bounded linear mapping $dJ_\phi : T_\phi L_{1,p}(M, N) = L_{1,p}(\phi^{-1}TN) \to \mathbf{R}$. Then as in (2.12) its norm is

$$\|dJ_\phi\| := \sup\{|dJ_\phi(u)|;\ u \in L_{1,p}(\phi^{-1}TN),\ \|u\|_{1,p} = 1\}.$$

LEMMA (3.8). *There exist positive constants $C_1$, $C_2$ satisfying the following:*

$$(d\mathcal{J}_{u_1} - d\mathcal{J}_{u_2})(u_1 - u_2) \geq C_1 \|u_1 - u_2\|_{1,p}{}^p - C_2 \|u_1 - u_2\|_p{}^p \tag{3.9}$$

*for $u_1$, $u_2 \in L_{1,p}(M, \mathbf{R}^K)$.*

Then putting

$$\psi(x) := C_1 x^{p-1}, \qquad \xi(x) := C_2 x^{p-1}, \qquad 0 < x < \infty,$$

(i) $\psi$ is strictly monotone increasing and $\psi(0) = 0$, $\lim_{x\to\infty} \psi(x) = \infty$. $\xi$ maps any bounded set of $\mathbf{R}^+ := \{x;\ x > 0\}$ into a bounded set of $\mathbf{R}^+$.
(ii) Moreover, (3.9) can be rewritten as

$$\begin{aligned}(d\mathcal{J}_{u_1} - d\mathcal{J}_{u_2})(u_1 - u_2) \geq \|u_1 - u_2\|_{1,p}\, \psi(\|u_1 - u_2\|_{1,p}) \\ - \|u_1 - u_2\|_p\, \xi(\|u_1 - u_2\|_p)\end{aligned} \tag{3.9$'$}$$

for $u_1$, $u_2 \in L_{1,p}(M, \mathbf{R}^K)$.

(3.10) We note that $L_{1,p}(M, N)$ is a bounded set in $L_p(M, \mathbf{R}^K)$. Therefore the convex closure of $L_{1,p}(M, N)$ in $L_p(M, \mathbf{R}^K)$ defined by

$$\begin{aligned} CH(L_{1,p}(M, N)) := \Bigg\{ \sum_{i=1}^{k} a_i u_i;\ u_i \in L_{1,p}(M, N), \\ 0 \leq a_i \leq 1\ (1 \leq i \leq k),\ \sum_{i=1}^{k} a_i = 1 \Bigg\}, \end{aligned}$$

is also bounded in $L_p(M, \mathbf{R}^K)$.

In fact, since $N$ is compact, $N \subset \mathbf{R}^K$ is a bounded set. Since $1 > \frac{m}{p}$, we get by Sobolev's Lemma (4.30) in Chapter 2, $L_{1,p}(M, N) \subset C^0(M, N)$. Thus, for any $u \in L_{1,p}(M, N)$,

$$\int_M |u|^p v_g \leq \sup_{x\in M} |u(x)|^p \operatorname{Vol}(M, g) < \infty,$$

which yields the conclusion. □

By (3.10) and Lemma (3.8), there exists a constant $B > 0$ such that

$$t \leq 2 \sup\{\|u\|_p;\ u \in CH(L_{1,p}(M, N))\} \Longrightarrow \xi(t) \leq B. \tag{3.11}$$

This follows from (i) of Lemma (3.8) and the inequality $\|u\|_p \leq \|u\|_{1,p}$ which follows from the definition. □

**Corollary (3.12).** *For any $u_1, u_2 \in CH(L_{1,p}(M, N))$,*

$$(d\mathscr{J}_{u_1} - d\mathscr{J}_{u_2})(u_1 - u_2) \geq \|u_1 - u_2\|_{1,p}(\psi(\|u_1 - u_2\|_{1,p}) - B). \tag{3.13}$$

**Proof.** For $u_1, u_2 \in CH(L_{1,p}(M, N))$,

$$\|u_1 - u_2\|_p \leq 2 \sup\{\|u\|_p;\ u \in CH(L_{1,p}(M, N))\}.$$

Thus, by (3.11), $\xi(\|u_1 - u_2\|_p) \leq B$, (3.9′) can be rewritten as (3.13). □

With these preparations, we prove the condition (C) in the following three steps. In order to prove the condition (C), we should show that:

> If a subset $S \subset L_{1,p}(M, N)$ satisfies that $J$ is bounded on $S$ and $\inf_S \|dJ\| = 0$. Then there exists a Cauchy sequence $\{\phi_i\}_{i=1}^{\infty}$ such that $\|dJ_{\phi_i}\| \to 0$ as $i \to \infty$.

Then since $L_{1,p}(M, N)$ is complete, there exists $\phi \in L_{1,p}(M, N)$ such that $\{\phi_i\}$ converges to $\phi$ in $L_{1,p}(M, N)$ and $dJ_\phi = 0$ since $\phi \mapsto \|dJ_\phi\|$ is continuous.

(The first step). *$S$ is a bounded set in $L_{1,p}(M, N)$.*

**Proof.** For any $u, a \in L_{1,p}(M, N)$, let $\sigma(t) := a + t(u - a)$, $0 \leq t \leq 1$, then $\sigma(t) \in L_{1,p}(M, \mathbf{R}^K)$. By (3.13) in Corollary (3.12), we get

$$(d\mathscr{J}_{\sigma(t)} - d\mathscr{J}_a)(t(u - a)) \geq t\,\|u - a\|_{1,p}(\psi(t\,\|u - a\|_{1,p}) - B). \tag{3.14}$$

Therefore, using (3.14), we get

$$\begin{aligned}
\mathscr{J}(u) &= \mathscr{J}(a) + \int_0^1 d\mathscr{J}_{\sigma(t)}(u - a)\,dt \\
&= \mathscr{J}(a) + d\mathscr{J}_a(u - a) + \int_0^1 \frac{1}{t}(d\mathscr{J}_{\sigma(t)} - d\mathscr{J}_a)(t(u - a))\,dt \\
&\geq \mathscr{J}(a) + d\mathscr{J}_a(u - a) + \|u - a\|_{1,p}\left\{\int_0^1 \psi(t\|u - a\|_{1,p})\,dt - B\right\}.
\end{aligned}$$

Thus, taking a positive constant $K$ as $K > \|d\mathscr{J}_a\|$, we get that

$$\mathscr{J}(u) \geq \mathscr{J}(a) + \|u - a\|_{1,p}\left\{\int_0^1 \psi(t\,\|u - a\|_{1,p})\,dt - (K + B)\right\}.$$

For any arbitrary $\epsilon > 0$, there exists $r > 0$ such that

$$\psi\left(\frac{r}{2}\right) > 2(1 + \epsilon)(K + B),$$

since $\lim_{x \to \infty} \psi(x) = \infty$. Moreover, if $\|u - a\|_{1,p} \geq r$, then we get

$$\begin{aligned}
\int_0^1 \psi(t\,\|u - a\|_{1,p})\,dt &\geq \int_{1/2}^1 \psi(tr)\,dt \geq \int_{1/2}^1 \psi(r/2)\,dt \\
&> \int_{1/2}^1 2(1 + \epsilon)(K + B)\,dt = (1 + \epsilon)(K + B),
\end{aligned}$$

since $\psi > 0$ is strictly monotone increasing. Thus, we obtain that

$$\|u-a\|_{1,p} \geq r \Longrightarrow \mathcal{J}(u) \geq \mathcal{J}(a) + \epsilon(K+B)\|u-a\|_{1,p}. \tag{3.15}$$

So we put $C := \sup_{\phi \in S} J(\phi) < \infty$ by the assumption that $J$ is bounded on $S$. Then if $u \in S$ satisfies $\|u-a\|_{1,p} \geq r$, by (3.15), we have

$$\|u-a\|_{1,p} \leq \frac{\mathcal{J}(u)-\mathcal{J}(a)}{\epsilon(K+B)} = \frac{J(u)-J(a)}{\epsilon(K+B)} \leq \frac{C-J(a)}{\epsilon(K+B)} < \infty.$$

That is, $S$ is included in the closed ball centered at $a$ with radius

$$\max\left\{r, \frac{C-J(a)}{\epsilon(K+B)}\right\}$$

in $L_{1,p}(M, \mathbf{R}^K)$. Thus, $S$ is bounded.

(The second step). Since $\inf_S \|dJ\| = 0$, we can take a sequence $\{\phi_i\}_{i=1}^\infty$ in $S$ such that $\|dJ_{\phi_i}\| \to 0$ as $i \to \infty$. By the first step, $\{\phi_i\}_{i=1}^\infty$ is a bounded set in $L_{1,p}(M, N)$, so by Lemma (3.7), we can choose a subsequence, denoted by the same letter, such that $d\mathcal{J}_{\phi_i}(\phi_i - \phi_j) \to 0$ as $i, j \to \infty$. Therefore, we obtain

$$(d\mathcal{J}_{\phi_i} - d\mathcal{J}_{\phi_j})(\phi_i - \phi_j) \to 0 \quad \text{as } i, j \to \infty. \tag{3.16}$$

(The third step). *The sequence $\{\phi_i\}_{i=1}^\infty$ in the second step contains a Cauchy subsequence.* (*Thus, we obtain the desired result, and the condition* (C).)

Proof. By the first step, $\{\phi_i\}_{i=1}^\infty$ is bounded in $L_{1,p}(M, N)$, so it contains a convergent subsequence, denoted by the same letter, in $L_p(M, \mathbf{R}^K)$ by Sobolev's Lemma (4.30) in Chapter 2. Thus, by (i) below Lemma (3.8),

$$\|\phi_i - \phi_j\|_p \xi(\|\phi_i - \phi_j\|_p) \to 0 \quad \text{as } i, j \to \infty.$$

This $\{\phi_i\}_{i=1}^\infty$ also satisfies (3.16), so by (3.9′) below Lemma (3.8), we obtain that

$$\|\phi_i - \phi_j\|_{1,p}\, \psi(\|\phi_i - \phi_j\|_{1,p}) \to 0 \quad \text{as } i, j \to \infty.$$

Thus, for any small given $\epsilon > 0$, there exists an $N > 0$ such that if $i, j \geq N$, then

$$\|\phi_i - \phi_j\|_{1,p}\, \psi(\|\phi_i - \phi_j\|_{1,p}) < \epsilon\, \psi(\epsilon),$$

since $\epsilon\, \psi(\epsilon) > 0$. Here since the function $x \mapsto x\, \psi(x)$ is strictly monotone increasing, we can conclude that

$$\|\phi_i - \phi_j\|_{1,p} < \epsilon$$

as desired. □

### 3.3. Proof of Lemma (3.8).

SUBLEMMA (3.17). *For $u = (u_1, \dots, u_K) \in C^\infty(M, \mathbf{R}^K)$, we put $F(du) := |du|^p$. Then the twice differentiation of $F$ at $du$ to the direction $dv$ satisfies*

$$d^2F_{du}(dv, dv) \geq p\,|dv|^{p-2}\,|dv|^2, \qquad u, v \in C^\infty(M, \mathbf{R}^K).$$

PROOF. Putting $p = 2\ell$,

$$\begin{aligned} d^2F_{du}(dv, dv) &= \frac{d^2}{dt^2}\bigg|_{t=0} (du + t\,dv, du + t\,dv)^\ell \\ &= 4\ell(\ell - 1)(du, du)^{\ell-2}(du, dv)^2 \\ &\quad + 2\ell(du, du)^{\ell-1}(dv, dv) \\ &\geq p\,|du|^{p-2}\,|dv|^2, \end{aligned}$$

since the first term of the last second of the right-hand is nonnegative. □

SUBLEMMA (3.18). *Let $V$ be a finite dimensional real vector space with a norm $|\ |$. Let $m$ be a positive integer. Then there exists a positive constant $C$ such that*

$$\int_0^1 |x + ty|^m\,dt \geq C\,|y|^m, \qquad x, y \in V.$$

PROOF. We may assume $y \neq 0$. Dividing the desired inequality by $|y|^m$, we should show

$$\int_0^1 \left|\frac{x}{|y|} + t\frac{y}{|y|}\right|^m dt \geq C,$$

that is, defining

$$F(x, y) := \int_0^1 |x + ty|^m \quad \text{for } |y| = 1,$$

we should show there exists a positive constant $C > 0$ such that $F(x, y) \geq C$ for all $x, y \in V$ with $|y| = 1$.

(i) For $|x| > 2$, we get

$$\int_0^1 |x + ty|^m\,dt \geq 1,$$

since $|x + ty| \geq 1$ for all $0 \leq t \leq 1$.

(ii) On the other hand, $F(x, y)$ is a positive continuous function on a compact set $\{(x, y) \in V \times V;\ |x| \leq 2,\ |y| = 1\}$. Because, if there were some $|x| \leq 2$ and $|y| = 1$ such that

$$\int_0^1 |x + ty|^m\,dt = 0,$$

then $|x + ty| = 0$ for all $0 \leq t \leq 1$. Then $x + ty = 0$ for all $0 \leq t \leq 1$. Since $|y| = 1$, we obtained $|x| = t$ for all $0 \leq t \leq 1$ which is a contradiction. Thus, $F$ attains a minimum $C' > 0$ on this compact set.

We may take the desired constant $C > 0$ as $C = \min\{1, C'\}$. □

(3.19) PROOF OF LEMMA (3.8). We may take $C_1 = C_2 = pC$, where $C$ is a constant in Sublemma (3.18) for $m = p - 2$.

Now for $u_1, u_2 \in L_{1,p}(M, \mathbf{R}^K)$, put

$$a := du_1(x), \quad b := du_2(x), \qquad x \in M,$$

and

$$c(t) := b + t(a - b), \qquad 0 \le t \le 1.$$

Then by Sublemma (3.17), we obtain

$$d^2F_{c(t)}(a - b, a - b) \ge p|c(t)|^{p-2}|a - b|^2 \tag{3.20}$$

and

$$\int_0^1 d^2F_{c(t)}(a - b, a - b) = \left[dF_{c(t)}(a - b)\right]_{t=0}^{t=1} = dF_a(a - b) - dF_b(a - b). \tag{3.21}$$

Now since $\mathcal{J}(u) = \int_M F(du)v_g$,

$$d\mathcal{J}_\phi(u) = \frac{d}{dt}\bigg|_{t=0} \mathcal{J}(\phi + tu) = \int_M \frac{d}{dt}\bigg|_{t=0} F(d\phi + t\,du)\,v_g = \int_M dF_{d\phi}(du)\,v_g.$$

Therefore, we obtain

$$\begin{aligned}(d\mathcal{J}_{u_1} - d\mathcal{J}_{u_2})(u_1 - u_2) &= \int_M (dF_{du_1} - dF_{du_2})(du_1 - du_2)\,v_g \\ &= \int_M \int_0^1 d^2F_{c(t)}(du_1 - du_2, du_1 - du_2)\,dt\,v_g \quad (\text{ by } (3.21)) \\ &\ge \int_M \int_0^1 p\,|c(t)|^{p-2}\,|du_1 - du_2|^2\,dt\,v_g \qquad (\text{by } (3.20)) \\ &\ge \int_M pC\,|du_1 - du_2|^{p-2}\,|du_1 - du_2|^2\,v_g \quad (\text{by Sublemma } (3.18)) \\ &= pC\,\|du_1 - du_2\|_p{}^p \qquad (\text{by definition of } \|\ \|_p) \\ &= pC\,(\|u_1 - u_2\|_{1,p}{}^p - \|u_1 - u_2\|_p{}^p) \quad (\text{by definition of } \|\ \|_{1,p})\end{aligned}$$

which implies the desired inequality. □

**3.4. Proof of Lemma (3.7).** Let $1 > \frac{m}{p}$, where $m = \dim(M)$. For $\phi \in L_{1,p}(M, N) \subset C^0(M, \mathbf{R}^K)$, define the mapping

$$P_\phi : L_{1,p}(M, \mathbf{R}^K) \ni u \mapsto P_\phi u \in T_\phi L_{1,p}(M, N)$$

by

$$(P_\phi u)(x) := P(\phi(x))\,u(x), \qquad x \in M. \tag{3.22}$$

Here note that

$$T_\phi L_{1,p}(M, N) = \{X \in L_{1,p}(M, \mathbf{R}^K);\ X(x) \in N_{\phi(x)},\ x \in M\},$$

and for $y \in N$, $P(y)$ is the projection of $\mathbf{R}^K$ onto the subspace $N_y$ with respect to the decomposition

$$\mathbf{R}^K = N_y \oplus N_y{}^\perp,$$

where $N_y := d\iota(T_y N)$ with the inclusion $\iota : N \subset \mathbf{R}^K$, and $N_y{}^\perp$ is the orthogonal complement of $N_y$ in $\mathbf{R}^K$. That is, for $a \in \mathbf{R}^K$, $P(y) = a^\top$, where

$$a = a^\top + a^\perp, \qquad a^\top \in N_y,\ a^\perp \in N_y{}^\perp.$$

Then we obtain

SUBLEMMA (3.23). *Let* $1 > \frac{m}{p}$, $m = \dim(M)$. *For all* $\phi \in L_{1,p}(M, N)$, *the mapping* $P_\phi : L_{1,p}(M, \mathbf{R}^K) \to T_\phi L_{1,p}(M, N) \subset L_{1,p}(M, \mathbf{R}^K)$ *is a surjective bounded linear projection. Moreover, the norm of this mapping defined by*

$$\|P_\phi\| := \sup\{\|P_\phi u\|_{1,p}/\|u\|_{1,p};\ 0 \neq u \in L_{1,p}(M, \mathbf{R}^K)\}$$

*maps a bounded set of* $L_{1,p}(M, N)$ *in* $\phi$ *to a bounded set.*

PROOF. Since $P(\phi(x))$ is the orthogonal projection of $\mathbf{R}^K$ onto a subspace $N_{\phi(x)}$, $P_\phi$ satisfies $P_\phi{}^2 = P_\phi \circ P_\phi = P_\phi$ and

$$P_\phi u = u \iff u(x) \in N_{\phi(x)}, \qquad x \in M$$
$$\iff u \in T_\phi L_{1,p}(M, N) = L_{1,p}(\phi^{-1}TN)$$

which implies the surjectivity of $P_\phi$.

Moreover the boundedness of $P_\phi$, i.e., $\|P_\phi u\|_{1,p} \le C\|u\|_{1,p}$ follows from the calculation of $\|P_\phi u\|_{1,p}$ by using

$$d_x(P_\phi u) = d_x(P(\phi(x))) \cdot u(x) + P(\phi(x)) \cdot d_x u.$$

The last claim follows from the continuity of the norm $\|\ \|$ whose calculation is left to the reader. □

(3.24) Moreover, for $\phi \in C^0(M, N)$, define by a $L(\mathbf{R}^K, \mathbf{R}^K)$-valued function $P(\phi)$ on $M$ by

$$P(\phi)(x) := P(\phi(x)) \in L(\mathbf{R}^K, \mathbf{R}^K), \qquad x \in M.$$

By Sobolev's Lemma (4.30) in Chapter 2, for $k > \frac{m}{p}$, with $m = \dim(M)$, $L_{k,p}(M, N) \subset C^0(M, N)$, and then

(i) if $\phi \in L_{k,p}(M, N)$, $P(\phi) \in L_{k,p}(M, L(\mathbf{R}^K, \mathbf{R}^K))$
and
(ii) the mapping $\phi \mapsto P(\phi)$ is continuous.

SUBLEMMA (3.25). *Let* $\{\phi_i\}_{i=1}^{\infty}$ *be a bounded sequence in* $L_{1,p}(M, \mathbf{R}^K)$. *Then there exists a subsequence, denoted by the same letter, such that for all* $C^\infty$*-vector fields* $X \in \mathfrak{X}(M)$,

$$X(P(\phi_i))(\phi_j - \phi_i) + (P(\phi_i) - P(\phi_j))(X\phi_j)$$

*converges to zero in* $L_p(M, \mathbf{R}^K)$ *as* $i, j \to \infty$.

PROOF. Let us take any small $0 < \epsilon < 1$ in such a way that $1 - \epsilon > \frac{m}{p}$, with $m = \dim(M)$. Then by Sobolev's Lemma (4.30) in Chapter 2, the inclusion $L_{1,p}(M, \mathbf{R}^K) \hookrightarrow L_{1-\epsilon,p}(M, \mathbf{R}^K)$ is completely continuous. Thus, a subsequence of $\{\phi_i\}_{i=1}^{\infty}$, denoted by the same letter, is convergent in $L_{1-\epsilon,p}(M, \mathbf{R}^K)$, say its limit $\phi_0$. Thus, by part (ii) in (3.24), in $L_{1-\epsilon,p}(M, L(\mathbf{R}^K, \mathbf{R}^K))$,

$$P(\phi_i) \to P(\phi_0) \quad \text{as } i \to \infty.$$

That is, $\|\phi_i - \phi_0\|_{1-\epsilon,p} \to 0$ and $\|P(\phi_i) - P(\phi_0)\|_{1-\epsilon,p} \to 0$ as $i \to \infty$. Here for all $X \in \mathfrak{X}(M)$, we have that

$$X(P(\phi_i)) \in L_p(M, L(\mathbf{R}^K, \mathbf{R}^K)) \quad \text{and} \quad X\phi_j \in L_p(M, \mathbf{R}^K),$$

and moreover,

$$\phi_j - \phi_i \in L_{1-\epsilon,p}(M, \mathbf{R}^K) \quad \text{and} \quad P(\phi_i) - P(\phi_j) \in L_{1-\epsilon,p}(M, L(\mathbf{R}^K, \mathbf{R}^K)).$$

Thus, we get

$$\begin{aligned}
&\|X(P(\phi_i))(\phi_j - \phi_i) + (P(\phi_i) - P(\phi_j))(X\phi_j)\|_p \\
&\quad \le C_1\left(\|X(P(\phi_i))\|_p \|\phi_j - \phi_i\|_{1-\epsilon,p} + \|P(\phi_i) - P(\phi_j)\|_{1-\epsilon,p} \|X\phi_j\|_p\right) \\
&\quad \le C_2\left(\|P(\phi_i)\|_{1,p} \|\phi_j - \phi_i\|_{1-\epsilon,p} + \|P(\phi_i) - P(\phi_j)\|_{1-\epsilon,p} \|\phi_j\|_{1,p}\right).
\end{aligned}$$

Since $\{\phi_j\}_{j=1}^{\infty}$ is bounded, by (ii) of (3.24), $\{\|P(\phi_i)\|_{1,p}\}_{i=1}^{\infty}$ is also a bounded set. Thus, the above converges to zero as $i, j \to \infty$. □

SUBLEMMA (3.26). *Assume that a sequence* $\{\phi_i\}_{i=1}^{\infty}$ *in* $L_{1,p}(M, N)$ *is bounded in* $L_{1,p}(M, \mathbf{R}^K)$. *Then there exists a subsequence, denoted by the same letter, in* $L_{1,p}(M, \mathbf{R}^K)$,

$$(I - P_{\phi_i})(\phi_i - \phi_j) \to 0 \text{ as } i, j \to \infty.$$

PROOF. (The first step) Since $P(\phi_i(x)) : \mathbf{R}^K \to N_{\phi_i(x)}$ is the projection, for any $u \in L_{1,p}(M, \mathbf{R}^K)$,

$$|(I - P(\phi_i(x)))u(x)| \le |u(x)|;$$

thus, we get

$$\|(I - P_{\phi_i})u\|_p \le \|u\|_p.$$

In particular, putting $u = \phi_i - \phi_j$,

$$\|(I - P_{\phi_i})(\phi_i - \phi_j)\|_p \le \|\phi_i - \phi_j\|_p.$$

On the other hand, since the inclusion $L_{1,p}(M, \mathbf{R}^K) \hookrightarrow L_p(M, \mathbf{R}^K)$ is colpletely continuous, a subsequence, denoted by the same letter, of $\{\phi_i\}_{i=1}^{\infty}$ is a Cauchy sequence of $L_p(M, \mathbf{R}^K)$. Thus, together with the above inequality, we get

$$\|(I - P_{\phi_i})(\phi_i - \phi_j)\|_p \to 0 \quad \text{as } i, j \to \infty. \tag{3.27}$$

Furthermore, we shall show in the following steps that for any $C^\infty$-vector field $X \in \mathfrak{X}(M)$,

$$X((I - P_{\phi_i})(\phi_i - \phi_j)) \to 0 \quad \text{as } i, j \to \infty, \tag{3.28}$$

in the space $L_p(M, \mathbf{R}^K)$. Then one may take a large enough $r \ge m = \dim(M)$ and $r$ $C^\infty$-vector fields on $M$, $\{X_1, \dots, X_r\}$, in such a way that at each point $x \in M$, each $v \in T_xM$ can be expressed as

$$v = \sum_{\ell=1}^{r} v_\ell (X_\ell)_x, \qquad v_\ell \in \mathbf{R},\ \ell = 1, \dots, r.$$

Then by (3.27), (3.28), and the defnition of $\| \ \|_{1,p}$, as $i, j \to \infty$,

$$\|(I - P_{\phi_i})\|_{1,p} \le \left\{ \|(I - P_{\phi_i})(\phi_i - \phi_j)\|_p + \sum_{\ell=1}^{r} \|X_\ell((I - P_{\phi_i})(\phi_i - \phi_j))\|_p \right\}^{1/p} \to 0$$

which implies the conclusion.

(The second step) To show (3.28), due to Sublemma (3.25), it suffices to see the following equation holds in $L_p(M, \mathbf{R}^K)$,

$$X((I - P_{\phi_i})(\phi_i - \phi_j)) = X(P(\phi_i))(\phi_j - \phi_i) + (P(\phi_i) - P(\phi_j))(X\phi_j).$$

But this can be shown as follows:

For $\phi \in C^\infty(M, N)$, $u \in C^\infty(M, \mathbf{R}^K)$, by definition of $P_\phi$ and $P(\phi)$, we have

$$X(P_\phi u) = X(P(\phi)u) = X(P(\phi))u + P(\phi)(Xu),$$

and both sides are continuous functions in $(\phi, u)$, the same equation holds for $\phi \in L_{1,p}(M, N)$, $u \in L_{1,p}(M, \mathbf{R}^K)$. Therefore, we get

$$\begin{aligned} X((I &- P_{\phi_i})(\phi_i - \phi_j)) \\ &= X(\phi_i - \phi_j) - X(P_{\phi_i}(\phi_i - \phi_j)) \\ &= X(\phi_i - \phi_j) - X(P(\phi_i))(\phi_i - \phi_j) - P(\phi_i)(X(\phi_i - \phi_j)) \\ &= X(P(\phi_i))(\phi_j - \phi_i) + (P(\phi_i) - P(\phi_j))(X\phi_j) \\ &\quad + \underline{P(\phi_j)(X\phi_j) + X(\phi_i - \phi_j) - P(\phi_i)(X\phi_i)}. \end{aligned}$$

Hence, if we show the underline should vanish, we get the desired equation.

(The third step) The underline coincides with

$$\{X\phi_i - P(\phi_i)(X\phi_i)\} - \{X\phi_j - P(\phi_j)(X\phi_j)\}.$$

Thus, it suffices to show that for all $\phi \in L_{1,p}(M, N)$ and $X \in \mathfrak{X}(M)$,

$$X\phi = P(\phi)(X\phi). \tag{3.29}$$

However, this can be seen as follows. Notice that

$$(X\phi)(x) = d\phi_x(X_x) \in T_{\phi(x)}N \cong N_{\phi(x)}, \qquad x \in M,$$

and $P(\phi)(x) = P(\phi(x))$ is the projection of $\mathbf{R}^K$ onto $N_{\phi(x)}$. Thus for any $\phi \in C^\infty(M, N)$,

$$(X\phi)(x) = P(\phi(x))(X\phi)(x), \qquad x \in M,$$

that is, $X\phi = P(\phi)(X\phi)$. In both sides of this equality, the mappings

$$\phi \mapsto X\phi, \qquad \phi \mapsto P(\phi)(X\phi)$$

are continuous from $L_{1,p}(M, N)$ into $L_p(M, N)$ and $C^\infty(M, N)$ is open and dense in $L_{1,p}(M, N)$. Thus, this equation holds for all $\phi \in L_{1,p}(M, N)$, and we obtain Sublemma (3.26). □

(3.30) Proof of Lemma (3.7). A $C^2$-function $J : L_{1,p}(M, N) \to \mathbf{R}$ is by definition $J = \mathcal{J}|_{L_{1,p}(M,N)}$ with a $C^2$- function $\mathcal{J} : L_{1,p}(M, \mathbf{R}^K) \to \mathbf{R}$. Then notice that for any $\phi \in L_{1,p}(M, N)$,

$$dJ_\phi = d\mathcal{J}_\phi\Big|_{T_\phi L_{1,p}(M,N)} \quad \text{and} \quad \|dJ_\phi\| \le \|d\mathcal{J}_\phi\|$$

by definition.

Now if a bounded sequence $\{\phi_i\}_{i=1}^\infty$ in $L_{1,p}(M, N)$ satisfying $\|dJ_{\phi_i}\| \to 0$ as $i \to \infty$, then

$$\begin{aligned} d\mathcal{J}_{\phi_i}(\phi_i - \phi_j) &= d\mathcal{J}_{\phi_i}(P_{\phi_i}(\phi_i - \phi_j)) + d\mathcal{J}_{\phi_i}((I - P_{\phi_i})(\phi_i - \phi_j)) \\ &= dJ_{\phi_i}(P_{\phi_i}(\phi_i - \phi_j)) + d\mathcal{J}_{\phi_i}((I - P_{\phi_i})(\phi_i - \phi_j)). \end{aligned}$$

Thus, by the definition of $\| \ \|$, we get

$$\begin{aligned} |d\mathcal{J}_{\phi_i}(\phi_i - \phi_j)| \le{}& \|dJ_{\phi_i}\| \, \|P_{\phi_i}\| \, \|\phi_i - \phi_j\|_{1,p} \\ &+ \|d\mathcal{J}_{\phi_i}\| \, \|(I - P_{\phi_i})(\phi_i - \phi_j)\|_{1,p}. \end{aligned} \tag{3.31}$$

Since $\{\|\phi_i\|_{1,p}\}_{i=1}^\infty$ is bounded, $\{\|\phi_i - \phi_j\|_{1,p}\}_{i=1}^\infty$ is also bounded. And the mapping

$$d\mathcal{J} : L_{1,p}(M, \mathbf{R}^K) \ni u \mapsto d\mathcal{J}_u \in L(L_{1,p}(M, \mathbf{R}^K), \mathbf{R})$$

sends a bounded set to a bounded set, so $\{d\mathcal{J}_{\phi_i}\}_{i=1}^\infty$ is also bounded. By Sublemma (3.23), $\{\|P_{\phi_i}\|\}_{i=1}^\infty$ is bounded. On the right-hand side of (3.31),

$\|d\mathscr{J}_{\phi_i}\| \to 0$ as $i \to \infty$ by the assumption, and by Sublemma (3.26), taking a subsequence, denoted by the same letter, we have

$$\|(I - P_{\phi_i})(\phi_i - \phi_j)\|_{1,p} \to 0 \quad \text{as } i, j \to \infty.$$

Therefore, we obtain

$$|d\mathscr{J}_{\phi_i}(\phi_i - \phi_j)| \to 0 \quad \text{as } i, j \to \infty,$$

which is the conclusion. □

Thus, we complete the proof of Theorem (3.3). All together with Theorem (2.17), Proposition (3.2), and Theorem (4.33) in Chapter 2, we obtain our main theorem:

THEOREM (3.32). *If* $1 > \frac{m}{p}$ *with* $m = \dim(M)$, *the* $C^2$*-function* $J$ *on* $L_{1,p}(M, N)$ *attains a minimum on each connected component.*

REMARK. A critical point of $J$ is called **p-harmonic** (cf. [Ha.L]). Theorem (3.32) says that if $1 > \frac{m}{p}$, then there exits a $p$-harmonic map in each homotopy class which minimizes. The existence, stability, and regularity of $p$-harmonic maps are very interesting unsolved problems. See Chapter 6.

## §4. An application to closed geodesics

In this section, we apply Theorem (3.32) to the theory of geodesics. Assume that $\dim(M) = 1$; that is, $M = S^1 = \mathbf{R}/\mathbf{Z}$, a circle. Let $(N, h)$ be an arbitrary compact Riemannian manifold. We also assume that by J. Nash's Theorem, $N$ is a closed submanifold of a large dimensional Euclidean space $\mathbf{R}^K$, and $h = \iota^* g_0$, where $\iota : N \subset \mathbf{R}^K$ is the inclusion and $g_0$ is the standard Riemannian metric on $\mathbf{R}^K$.

Note that any element in $C^0(S^1, N)$ can be regarded as a continuous curve of the closed interval $[0, 1]$ into $N$ with the period 1.

Now let $p = 2$. Since $m = \dim(M) = \dim(S^1) = 1$, $\frac{m}{p} = \frac{1}{2} < 1$. Notice that $L_{1,2}(S^1, N)$ is a closed submanifold of the Hilbert space $L_{1,2}(S^1, \mathbf{R}^K)$. By Sobolev's Lemma (4.30) in Chapter 2, we get

$$L_{1,2}(S^1, N) \subset C^0(S^1, N), \qquad L_{1,2}(S^1, \mathbf{R}^K) \subset C^0(S^1, \mathbf{R}^K).$$

For each point $\sigma \in L_{1,2}(S^1, N)$, the tangent space of $L_{1,2}(S^1, N)$,

$$T_\sigma L_{1,2}(S^1, N) = L_{1,2}(\sigma^{-1}TN)$$

is given by

$$T_\sigma L_{1,2}(S^1, N) = \{X \in L_{1,2}(S^1, \mathbf{R}^K);\ X(t) \in T_{\sigma(t)}N, \\ \text{for all } t \in [0, 1],\ X(0) = X(1)\}.$$

The inner product $(\ ,\ )_1$ as a Hilbert space of $L_{1,2}(S^1, \mathbf{R}^K)$ is given by

$$(\sigma, \rho)_1 := \int_0^1 (\sigma'(t), \rho'(t))\, dt + \int_0^1 (\sigma(t), \rho(t))\, dt,$$

for $\sigma, \rho \in L_{1,2}(S^1, \mathbf{R}^K)$. Here writing $\sigma(t) = (\sigma_1(t), \dots, \sigma_K(t))$, and $\rho(t) = (\rho_1(t), \dots, \rho_K(t))$, we define

$$(\sigma'(t), \rho'(t)) := \sum_{A=1}^{K} \sigma_A{}'(t)\, \rho_A{}'(t), \qquad (\sigma(t), \rho(t)) := \sum_{A=1}^{K} \sigma_A(t)\, \rho_A(t).$$

The Riemannian metric $g_0$ on the Hibert space $L_{1,2}(S^1, \mathbf{R}^K)$ is given as follows: At each point $\sigma \in L_{1,2}(S^1, \mathbf{R}^K)$, since $T_\sigma L_{1,2}(S^1, \mathbf{R}^K) = L_{1,2}(S^1, \mathbf{R}^K)$, for all $u, v \in L_{1,2}(S^1, \mathbf{R}^K)$,

$$g_0(u, v) := (u, v)_1.$$

The distance of $L_{1,2}(S^1, \mathbf{R}^K)$ induced from $g_0$, $\rho_0$, is

$$\rho_0(\sigma, \rho) = \|\sigma - \rho\|_{1,2} = (\sigma - \rho, \sigma - \rho)_1{}^{1/2}, \qquad \sigma, \rho \in L_{1,2}(S^1, \mathbf{R}^K),$$

and then $(L_{1,2}(S^1, \mathbf{R}^K), g_0)$ is a complete Hilbert manifold.

Since $L_{1,2}(S^1, \mathbf{R}^K)$ is a closed submanifold of a Hilbert space $L_{1,2}(S^1, \mathbf{R}^K)$, we can give a Riemannian metric $g$ by $g := \iota^* g_0$, where

$$\iota : L_{1,2}(S^1, N) \subset L_{1,2}(S^1, \mathbf{R}^K)$$

is the inclusion. This Riemannian metric $g$ is for all $\sigma \in L_{1,2}(S^1, N)$, $X$, $Y \in T_\sigma L_{1,2}(S^1, N)$,

$$g_\sigma(X, Y) = \int_0^1 (X'(t), Y'(t))\, dt + \int_0^1 (X(t), Y(t))\, dt.$$

As the same way of Proposition (3.2), $(L_{1,2}(S^1, N), g)$ is a complete $C^\infty$-Riemannian manifold. The function $J$ on $L_{1,2}(S^1, N)$ is

$$J(\sigma) := \int_0^1 (\sigma'(t), \sigma'(t))\, dt = \int_0^1 h_{\sigma(t)}(\sigma'(t), \sigma'(t))\, dt,$$

where $h$ is a Riemannian metric on $N$. Then applying Theorem (3.32), we find there exists an element $\gamma$ in each connected component of $L_{1,2}(S^1, N)$ which minimizes $J$. As we shall show in the next chapter, each critical point of $J$ is a geodesic and it turns out that it is smooth, the above $\gamma$ is a periodic geodesic, called a **closed geodesic**. Thus, we obtain a well-known theorem:

THEOREM (4.2). *For any compact $C^\infty$-Riemanniann manifold $(N, h)$, in each element in the fundamental group (cf. (4.45) in Chapter 2) $\pi_1(N)$ of $N$ there exists a closed geodesic which attains a minimum of $J$ in the homotopy class.*

REMARK 1. The function $J$ of $L_{1,2}(S^1, N)$ in (4.1) is called the **energy** or the **action integral**. In the next chapter, we shall denote it by $E$. For a geodesic, in particular, for a closed geodesic, see [K1], [K2].

REMARK 2. Theorems (3.3), (3.32) cannot be applied to the theory of harmonic maps in Chapter 4 because of the crucial condition " $1 > \frac{p}{m}$, $m = \dim(M)$ ". However, these theorems are still fundamental and essential to show the existence of harmonic mappings. The existence theory can be treated also in Chapter 6, and for more detail see [S.Uh], [Uh 2], [M.M]. Overcoming the condition (C) is still one of the most important problems in differential geometry and analysis.

**≪ Coffee Break ≫ The isoperimetric problem and Queen Dido**

The isoperimetric problem is the problem of finding the maximum area of a plain domain enclosed by a simple closed curve with a given length. The ancient Greeks seemed already to know the answer that only a disc achieves a maximum area. This problem is one of the origins of the calculus of the variations. To solve this problem is to show that letting $L$, be the length of the perimeter of a domain and letting $A$, be its area,

$$4\pi A \leq L^2 \quad \text{(the isoperimetric inequality)},$$

and the equality holds if and only if the domian is a disc. To give a rigorous proof is rather new, and H. A. Schwarz gave the first proof in 1890. Here we introduce J. Steiner's idea called "symmetrization" to give the proof which appeared in J. Reine Angew Math. 24 (1842), 93–152.

Steiner's idea of the proof is as follows: Let $D$ be a domain and let $C$ be its smooth simple closed boundary curve. Take an arbitrary line $\ell$. Draw perpendicular lines toward the given line $\ell$ to make the following domain $D_\ell$ which is symmetric with respect to $\ell$.

Take a perpendicular line $\ell'$ to $\ell$. Let $M$ be the intersection of $\ell$ and $\ell'$. Choose two points $M_1$, $M_2$ on the line $\ell'$ in such a way that $M$ is the midpoint of the segment between $M_1$ and $M_2$ with the length equal to that of the intersection of $\ell'$ and $D$. Continuing this process to all perpendicular lines $\ell'$, we get $D_\ell$ (see Figure 3.4).

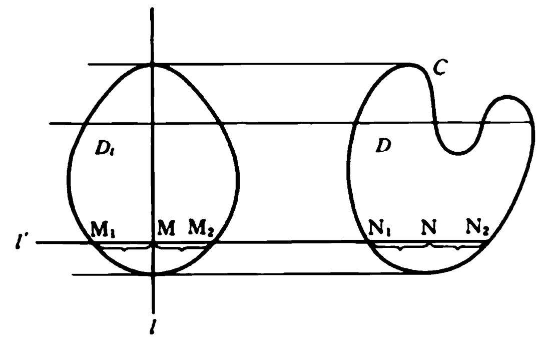



FIGURE 3.4

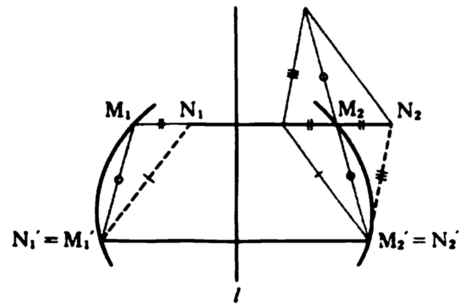



FIGURE 3.5

Then comparing the areas and the length of $D_\ell$ and these of the original domain $D$, denoting by $A_\ell$, $A$ and $L_\ell$, $L$ the corresponding areas and lengths to $D_\ell$, $D$, respectively, we get

$$A = A_\ell, \qquad L \geq L_\ell.$$

Because $A = A_\ell$ by definition of area, and for the lengths of boundaries, it suffices to show

$$2\overline{M_2M_2'} \leq \overline{N_1N_1'} + \overline{N_2N_2'},$$

if we assume

$$\overline{M_1M_2} = \overline{N_1N_2} \quad \text{and} \quad \overline{M_1'M_2'} = \overline{N_1'N_2'},$$

for two trapezoids $N_1N_1'N_2'N_2$ and $M_1M_1'M_2'M_2$ with $\overline{M_1M_1'} = \overline{M_2M_2'}$. See Figure 3.5.

Now drawing another line $\ell'$ and doing the same procedure to $D_\ell$, we get a domain $D_{\ell\ell'}$ which has a more symmetry, the same area $A_{\ell\ell'} = A$ and a shorter length $L_{\ell\ell'}$ for the boundary. Continuing this process repeatedly, we finally reach to a disc $D_0$. Denoting by $A(D_0)$, $L(D_0)$, the area and length of the boundary of the disc $D_0$

$$\begin{cases} A = A_\ell = A_{\ell\ell'} = \cdots = A(D_0), \\ L \geq L_\ell \geq L_{\ell\ell'} \geq \cdots \geq L(D_0). \end{cases}$$

But for the disc $D_0$, $4\pi A(D_0) = L(D_0)^2$, so we obtain finally the desired inequality:

$$4\pi A \leq L^2.$$

These process is the main idea of Steiner's symmetrization, but there is the problem of reaching the disc $D_0$.

If we consider the same problem near a seaside with a straight coastline, the answer is a semicircle faced toward the coast line. In fact, let us give a curve $C$ whose endpoints touch the coast, and a mirror curve $C'$ reflecting it with respect to the coast, we get a simple closed curve $CC'$. Denote by $A_C$ the area of the domain enclosed by $C$ and the coast, and let $L$ be the length of $C$. Then applying the isoperimetric inequality to a domain enclosed by $CC'$, we get the inequality:

$$4\pi(2A_C) \leq (2L)^2,$$

the equality holds if and only if $C$ is a semicircle. Then for a given $L$, the maximal area is $L^2/(2\pi)$. See Figure 3.6.

Now the "Aeneid" by Virgil ( 70–19 B.C. an ancient Roman poet ) tells us a famous story of Queen Dido of ancient Carthage:

> Dido was a princess of ancient Phoenicia who lived at Tyre. But her brother, Pygmalion, killed her husband and took the position of King. She escaped from Tyre by getting on board a boat with her followers and reached land in 900 B.C. at a place later named Carthage. To build their own country, they were willing to buy land from King Jarbas at Numidia who governed there. But the King did not like this and promised only to sell a tiny bit of land of the same area as could be enclosed by the spread out skin of a cow. In order to make full use of this condition, Dido extended the interpretation and let her follower cut a cow's skin into thin threads and connect them making a long string with length almost 1 km (0.62 mile). Using this string, they enclosed a land by a semi-circle from the coast of the Mediterranean Sea. She enclosed a land area of almost 16 hectare (40 acres).

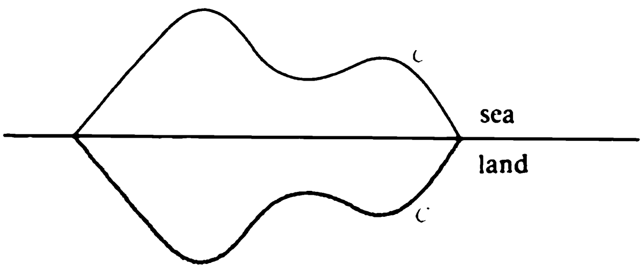



FIGURE 3.6

CHAPTER 4

# Harmonic Mappings

It seems the theory of harmonic mappings was started with a study of J. Eells in 1958 which showed the space of mappings becomes an infinite dimensional manifold and with the natural idea of asking for a mapping which is critical for some function (called the energy or the action integral) on the space. This is the work of J. Eells and J. H. Sampson, in 1964.

In this chapter, we derive the first derivative of the energy (called the first variational formula). We explain the notion of harmonic mappings (or the nonlinear sigma model) and give several examples. Readers can start with this chapter independent of the previous chapters. The notions and notations used in this chapter are all explained.

## §1. What is a harmonic mapping?

In this section, we define the quantity called the energy or the action integral for any smooth mapping between two compact Riemannian manifolds $(M, g)$, $(N, h)$, and we define a harmonic mapping as its critical point. We calculate the first derivative of the energy and derive the Euler-Lagrange equation. Solutions of this equation are harmonic mappings.

In the calculations of this chapter, we always use the following notations. We put

$$\dim(M) = m, \qquad \dim(N) = n.$$

We often embed $(N, h)$ isometrically into the Euclidean space $\mathbf{R}^K$ of a sufficiently large dimension $K$. That is, $N$ is a closed submanifold of the Euclidean space $\mathbf{R}^K$, and the Riemannian metric $h$ is the pull back of the standard metric $g_0$ on $\mathbf{R}^K$ by the inclusion $\iota : N \subset \mathbf{R}^K$,

$$h = \iota^* g_0.$$

We use the subscripts such as

$$1 \le i, j, \dots, \le m, \qquad 1 \le \alpha, \beta, \dots, \le n, \qquad 1 \le A, B, \dots, \le K.$$

The Levi-Civita connections and the curvature tensors of $(M, g)$, $(N, h)$, $(\mathbf{R}^K, g_0)$ are denoted, respectively, by

$$\nabla, \quad {}^N\nabla, \quad \nabla^0; \qquad R, \quad {}^N R, \quad R^0.$$

### 1.1. The energy (action integral).

(1.1) *The energy density function.* For a $C^1$-mapping $\phi \in C^1(M, N)$, we define the **energy density function** of $\phi$, $e(\phi) \in C^0(M)$, by

$$e(\phi) = \frac{1}{2}\mathrm{Tr}_g(\phi^*h)(x) = \frac{1}{2}\sum_{i=1}^{m}(\phi^*h)(u_i, u_i) \qquad (1.2)$$
$$= \frac{1}{2}\sum_{i=1}^{m} h(\phi_* u_i, \phi_* u_i), \qquad x \in M,$$

where $\phi^*h$ is the pull back of $h$ by $\phi$, $\mathrm{Tr}_g(\phi^*h)$ is the trace of a tensor field $\phi^*h$ by $g$ of which the definition is twice the right-hand side of (1.2). We denote by $\phi_* : T_xM \to T_{\phi(x)}N$, the differentiation of $\phi$. Here $\{u_i\}_{i=1}^m$ is an orthonormal basis of the tangent space $T_xM$ at $x$ with respect to $g_x$. (1.2) does not depend on a choice of $\{u_i\}_{i=1}^m$.

The continuity of $e(\phi)$ and that $e(\phi) \in C^\infty(M)$ if $\phi \in C^\infty(M, N)$ can be seen as follows:

We extend an orthonormal basis $\{u_i\}_{i=1}^m$ of $(T_xM, g_x)$ at $x \in M$ to $m$ $C^\infty$-vector fields $\{e_i\}_{i=1}^m$ on a coordinate neighborhood $U$ at $x$ such that at each point $z \in U$, $\{e_i(z)\}_{i=1}^m$ is also an orthonormal basis of $(T_zM, g_z)$ as follows. We call such $\{e_i\}_{i=1}^m$ a **locally defined orthonormal frame field** on $U$. Take a coordinate $(x_1, \dots, x_n)$ on $U$ around $x$ in such a way that $(\frac{\partial}{\partial x_i})_x = u_i$, $1 \le i \le m$, (see §3 in Chapter 2), and orthonormalize $\{(\frac{\partial}{\partial x_i})_z\}_{i=1}^m$ into $\{e_i\}_{i=1}^m$ by Schmidt process at each point $z$. Then by means of the form, $e_i$ are $C^\infty$ on $U$. We may give an alternative proof as we take a small neighborhood $U$ such that each point $z$ can be connected by a unique geodesic emanating from $x$. Then we define $e_i(z)$ to be the parallel transport of $u_i$ along such a geodesic from $x$ to $z$. Then using $\{e_i\}_{i=1}^m$ on $U$,

$$e(\phi) = \frac{1}{2}\sum_{i=1}^{m}(\phi^*h)(e_i, e_i),$$

the right-hand side of which is a continuous function, or $C^\infty$ on $U$ if $\phi \in C^\infty(M, N)$.

Taking local coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$ on neighborhoods of $x$, $\phi(x)$ in $M$, $N$, respectively, and putting $\phi^\alpha := y_\alpha \circ \phi$, $\alpha = 1, \dots, n$, then (1.2) can be expressed as

$$e(\phi)(z) = \frac{1}{2}\sum_{i,j,\alpha,\beta} g^{ij}(z)h_{\alpha\beta}(\phi(z))\frac{\partial\phi^\alpha}{\partial x_i}(z)\frac{\partial\phi^\beta}{\partial x_j}(z), \qquad (1.3)$$

at each point $z$ in the neighborhood of $x$. Here $(g^{ij})$ is the inverse matrix of $(g_{ij})$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, and $h_{\alpha\beta} = h(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta})$.

(1.4) *Alternative expression.* If $\iota : N \subset \mathbf{R}^K$, a closed submanifold and $h = \iota^* g_o$, then $e(\phi)$ can be expressed as

$$e(\phi) = \frac{1}{2}|d\phi|^2 = \frac{1}{2}\sum_{A=1}^{K}|d\phi_A|^2. \tag{1.5}$$

Here we explain the notations. We denote

$$\phi(x) = \iota \circ \phi(x) = (\phi_1(x), \ldots, \phi_K(x)) \in \mathbf{R}^K, \qquad x \in M,$$

then $\phi_A \in C^1(M)$ and $d\phi_A$, $1 \le A \le K$ are $K$ 1-forms on $M$. $|\ |$ is the norm on $T^*M$ induced from the Riemannian metric $g$ on $M$.

(1.5) can be proved as follows: Since $h = \iota^* g_0$ and the pull back $\phi^* g_0$ of $\phi = \iota \circ \phi$ is given by

$$\phi^* g_0 = \sum_{A=1}^{K} d\phi_A \otimes d\phi_A,$$

we get

$$\sum_{i=1}^{m}(\phi^* h)(e_i, e_i) = \sum_{i=1}^{m}\sum_{A=1}^{K} d\phi_A(e_i)\, d\phi_A(e_i) = \sum_{A=1}^{K}|d\phi_A|^2.$$

Here we used the equation $|d\phi_A|^2 = \sum_{i=1}^m d\phi_A(e_i)\, d\phi_A(e_i)$ which follows by the definition of $|\ |$ on $T^*M$.

The expression (1.5) is useful because it is available everywhere on $M$ which was used also in Chapter 3.

(1.6) *The energy or the action integral.* For $\phi \in C^1(M, N)$, the integral

$$E(\phi) := \int_M e(\phi)\, v_g = \frac{1}{2}\int_M |d\phi|^2\, v_g$$

is called the **energy** or the **action integral** of $\phi$.

REMARK. As in Chapters 2 and 3, let $L_{1,p}(M, \mathbf{R}^K)$ be the Banach space completion of $C^1(M, \mathbf{R}^K)$ by means of the norm $\|\ \|_{1,p}$ defined by

$$\|\phi\|_{1,p} := \left(\int_M |d\phi|^p\, v_g + \int_M |\phi|^p\, v_g\right)^{1/p}$$

for all $p > 0$. If $1 > \frac{m}{p}$, Sobolev's Lemma (4.30) in Chapter 2, says that $L_{1,p}(M, \mathbf{R}^K) \subset C^0(M, \mathbf{R}^K)$, so

$$L_{1,p}(M, N) := \{\phi \in L_{1,p}(M, \mathbf{R}^K);\ \phi(x) \in N,\ \forall x \in M\}$$

can be well defined. On the other hand, regarding the definition (1.6) of $E(\phi)$, since

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, v_g,$$

and taking $p = 2$, it seems natural for $E$ to be a function on

$$L_{1,2}(M, N) := \{\phi \in L_{1,2}(M, \mathbf{R}^K);\ \phi(x) \in N,\ \text{a.e. } x \in M\}.$$

Here one meets some problem on the constrain condition: "$\phi(x) \in N$ a.e. $x \in M$". $L_{1,2}(M, N)$ is only a subset of $L_{1,2}(M, \mathbf{R}^K)$, and it is difficult to give a definition for critical points of $E$. In general, if $\dim(M) \geq 3$, it is known that $L_{1,2}(M, N) \cap C^0(M, N)$ is not dense in $L_{1,2}(M, N)$. See R. M. Schoen [Sc] for more detail. In this section, we avoid these troublesome points, and shall treat only $C^\infty(M, N)$ without further comment.

### 1.2. Definition of harmonic mappings.

DEFINITION (1.7). We call $\phi \in C^\infty(M, N)$ a **harmonic mapping** (or **a nonlinear sigma model**) if $\phi$ is a critical point of $E$ at $C^\infty(M, N)$, i.e., for any smooth variation $\phi_t \in C^\infty(M, N)$ with $-\epsilon < t < \epsilon$ of $\phi$, we have

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0.$$

Here a **smooth variation** $\phi_t$ means that $\phi_0 = \phi$, and $\phi_t$ depends on $t$ of class $C^\infty$; that is, the mapping $F : (-\epsilon, \epsilon) \times M \to N$ defined by

$$F(t, x) := \phi_t(x), \qquad -\epsilon < t < \epsilon, \ x \in M,$$

satisfies

$$\begin{cases} F(0, x) = \phi(x), & x \in M, \text{ and} \\ F : (-\epsilon, \epsilon) \times M \to N, & \text{a } C^\infty\text{-mapping.} \end{cases}$$

(1.8) *Variation vector fields.* For any smooth variation $\phi_t$ of $\phi$, $-\epsilon < t < \epsilon$, putting

$$V(x) := \left.\frac{d}{dt}\right|_{t=0} \phi_t(x), \qquad x \in M, \tag{1.9}$$

then $V$ is a $C^\infty$-mapping of $M$ into the tangent bundle $TN$ satisfying

$$V(x) \in T_{\phi(x)}N, \qquad x \in M. \tag{1.10}$$

Conversely, for any $C^\infty$-mapping $V : M \to TN$ satisfying (1.10), defining

$$\phi_t(x) := \exp_{\phi(x)}(t\,V(x)), \qquad x \in M,$$

we see that $\phi_t \in C^\infty(M, N)$ and

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = V(x), \qquad x \in M$$

by definition. Such a vector field $V$ is called a **variation vector field** along $\phi$. See Figure 4.1.

Variation vector fields can be regarded as sections of some vector bundle over $M$ in the following way:

Let $\phi^{-1}TN$ be the **induced bundle** over $M$ of $TN$ by $\phi$, that is, let $\pi : TN \to N$, the projection, and let

$$\begin{aligned} \phi^{-1}TN &= \{(x, u) \in M \times TN;\ \pi(u) = \phi(x),\ x \in M\} \\ &= \bigcup_{x \in M} T_{\phi(x)}N, \end{aligned}$$

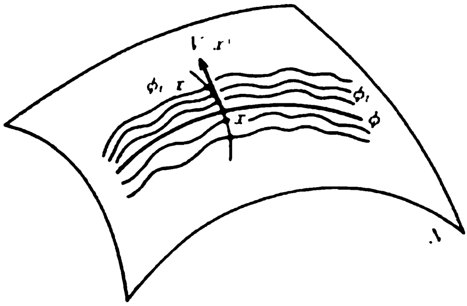


FIGURE 4.1

then the set of all $C^\infty$-sections of $\phi^{-1}TN$, denoted by $\Gamma(\phi^{-1}TN)$, becomes the set of all variation vector fields:

$$\Gamma(\phi^{-1}TN) = \{V;\ M \to TN,\ C^\infty\text{-mappings},\ V(x) \in T_{\phi(x)}N,\ x \in M\}.$$

As $L_{1,p}(\phi^{-1}TN)$ is identified with the tangent space $T_\phi L_{1,p}(M, N)$ of a manifold $L_{1,p}(M, N)$ (Theorem (4.33) in Chapter 2), the set of all variation vector fields, $\Gamma(\phi^{-1}TN)$, can be regarded as a tangent space $T_\phi C^\infty(M,N)$ of a "manifold" $C^\infty(M,N)$.

**1.3. The induced connection on the induced bundle.** In order to derive the first variation formula, we prepare some notions.

(1.11) *Connections of a vector bundle.* In general, let $E$ be an arbitrary $C^\infty$-vector bundle over $M$,

$$E = \bigcup_{x\in M} E_x,$$

where $\pi : E \to M$ is the projection and $E_x = \pi^{-1}(x)$ is a vector space of dimension $r$ (not depending on $x \in M$), called the **fiber** at $x$. We denote by $\Gamma(E)$, the set of all $C^\infty$-sections of $E$, that is, the set of all $C^\infty$-mappings $s : E \to M$ such that

$$\pi \circ s = \text{id}, \quad \text{that is, } s(x) \in E_x,\ x \in M.$$

Then $\Gamma(E)$ is a vector space and a $C^\infty(M)$-module; that is, for all $f \in C^\infty(M)$, $s$, $s_1$, $s_2 \in \Gamma(E)$, if we define

$$(f\cdot s)(x) := f(x)s(x), \qquad (s_1+s_2)(x) := s_1(x)+s_2(x), \qquad x \in M,$$

then $f\cdot s$, $s_1+s_2 \in \Gamma(E)$.

Then $\tilde{\nabla}$ is a **connection** (or **covariant differentiation**) of $E$ if for all $C^\infty$-vector field $X \in \mathfrak{X}(M)$, a mapping

$$\tilde{\nabla}_X : \Gamma(E) \ni s \mapsto \tilde{\nabla}_X s \in \Gamma(E)$$

satisfies the following conditions: for all $f \in C^\infty(M)$, $X$, $Y \in \mathfrak{X}(M)$, $s$, $s_1$, $s_2 \in \Gamma(E)$,

(i) $\tilde{\nabla}_{X+Y} s = \tilde{\nabla}_X s + \tilde{\nabla}_Y s$,

(ii) $\tilde{\nabla}_{fX} s = f\tilde{\nabla}_X s$,

(iii) $\tilde{\nabla}_X(s_1 + s_2) = \tilde{\nabla}_X s_1 + \tilde{\nabla}_X s_2$,

(iv) $\tilde{\nabla}_X(fs) = Xf \cdot s + f\tilde{\nabla}_X s$.

By (ii), $(\tilde{\nabla}_X s)(x) \in X_x$, $x \in M$ is uniquely determined only on $u = X_x \in T_xM$, so it can be written as $\tilde{\nabla}_u s = (\tilde{\nabla}_X s)(x)$.

For $X$, $Y \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, we let

$$R^{\tilde{\nabla}}(X, Y)s := \tilde{\nabla}_X(\tilde{\nabla}_Y s) - \tilde{\nabla}_Y(\tilde{\nabla}_X s) - \tilde{\nabla}_{[X, Y]} s.$$

$R^{\tilde{\nabla}}$ is called the **curvature tensor field** of $E$ relative to $\tilde{\nabla}$. We have

$$R^{\tilde{\nabla}}(fX, gY)s = fg\,R^{\tilde{\nabla}}(X, Y)s, \qquad f, g \in C^\infty(M).$$

Next a vector bundle $E$ admits an inner product $h$ if each fiber $E_x$ has an inner product

$$h_x : E_x \times E_x \to \mathbf{R},$$

and $h_x$ depends on $x$ smoothly of class $C^\infty$; that is, for any $s_1$, $s_2 \in \Gamma(E)$, the function $h(s_1, s_2)$ on $M$ defined by $h(s_1, s_2)(x) := h_x(s_1(x), s_2(x))$, $x \in M$ is $C^\infty$. Moreover, a connection $\tilde{\nabla}$ is **compatible with** an inner product $h$ of $E$ if for each $X \in \mathfrak{X}(M)$, $s_1$, $s_2 \in \Gamma(E)$,

$$X \cdot h(s_1, s_2) = h(\tilde{\nabla}_X s_1, s_2) + h(s_1, \tilde{\nabla}_X s_2). \tag{1.12}$$

(1.13) *The induced connection.* Let us denote by $\nabla$, ${}^N\nabla$, the Levi-Civita connections on $(M, g)$, $(N, h)$. Then for $\phi \in C^\infty(M, N)$, we can define the **induced connection** $\tilde{\nabla}$ on the induced bundle $E = \phi^{-1}TN = \bigcup_{x\in M} T_{\phi(x)}N$ as follows:

For $X \in \mathfrak{X}(M)$, $V \in \Gamma(\phi^{-1}TN)$, define $\tilde{\nabla}_X V \in \Gamma(\phi^{-1}TN)$ by

$$(\tilde{\nabla}_X V)(x) = {}^N\nabla_{\phi_* X} V := \frac{d}{dt}\bigg|_{t=0} {}^NP_{\phi\circ\sigma_t}{}^{-1} V(\sigma(t)), \qquad x \in M, \tag{1.14}$$

where $t \mapsto \sigma(t) \in M$ is a $C^1$-curve in $M$ satisfying $\sigma(0) = x$, $\sigma'(0) = X_x \in T_xM$, and $\sigma_t$ is a curve given by $\sigma_t(s) := \sigma(s)$, $0 \le s \le t$, i.e., the restriction of $\sigma$ to the part between $x$ and $\sigma(t)$. ${}^NP_{\phi\circ\sigma_t} : T_{\phi(x)}N \to T_{\phi(\sigma(t))}N$ is the parallel transport along a $C^1$-curve $\phi \circ \sigma_t$ with respect to the Levi-Civita connection ${}^N\nabla$ on $(N, h)$.

Indeed, that $\tilde{\nabla}$ is a connection on $E = \phi^{-1}TN$ follows from checking (i)–(iv). For example, (iv) says that for $f \in C^\infty(M)$, $V \in \Gamma(\phi^{-1}TN)$,

$$\begin{aligned}\tilde{\nabla}_X(fV)(x) &= \frac{d}{dt}\Big|_{t=0} {}^N P_{\phi\circ\sigma_t}{}^{-1}(f(\sigma(t))\,V(\sigma(t)))\\ &= \left\{\frac{d}{dt}\Big|_{t=0} f(\sigma(t))\right\} V(x) + f(x)\left\{\frac{d}{dt}\Big|_{t=0} {}^N P_{\sigma\circ\sigma_t}{}^{-1}\,V(\sigma(t))\right\}\\ &= X_x f \cdot V(x) + f(x)\,(\tilde{\nabla}_X V)(x), \qquad x \in M.\end{aligned}$$

The other properties can be shown by similar arguments. See Figure 4.2.

Furthermore, $E = \phi^{-1}TN$ admits an inner product from the Riemannian metric $h$ on $N$, denoted by the same letter $h$, by

$$h_{\phi(x)} : T_{\phi(x)}N \times T_{\phi(x)}N \to \mathbf{R},$$

since $E_x = T_{\phi(x)}N$. The above induced connection $\tilde{\nabla}$ is compatible with this inner product $h$. In fact, for $V_1$, $V_2 \in \Gamma(\phi^{-1}TN)$, $X \in \mathfrak{X}(M)$, $x \in M$,

$$\begin{aligned}X_x h(V_1, V_1) &= \frac{d}{dt}\Big|_{t=0} h_{\phi(\sigma(t))}(V_1(\sigma(t)), V_2(\sigma(t)))\\ &= \frac{d}{dt}\Big|_{t=0} h_{\phi(x)}({}^N P_{\sigma\circ\sigma_t}{}^{-1}\,V_1(\sigma(t)), {}^N P_{\sigma\circ\sigma_t}{}^{-1}\,V_2(\sigma(t)))\\ &= h_{\phi(x)}\left(\frac{d}{dt}\Big|_{t=0} {}^N P_{\sigma\circ\sigma_t}{}^{-1}\,V_1(\sigma(t)), V_2(x)\right)\\ &\quad + h_{\phi(x)}\left(V_1(x), \frac{d}{dt}\Big|_{t=0} {}^N P_{\sigma\circ\sigma_t}{}^{-1}\,V_2(\sigma(t))\right)\\ &= h_{\phi(x)}(\tilde{\nabla}_{X_x} V_1, V_2) + h_{\phi(x)}(V_1, \tilde{\nabla}_{X_x} V_2),\end{aligned}$$

where we used the fact that ${}^N P_{\phi\circ\sigma_t} : (T_{\phi(t)}N, h_{\phi(x)}) \to (T_{\phi(\sigma(t))}, h_{\phi(\sigma(t))})$ is an isometry.

(1.14′) *Examples of elements of* $\Gamma(\phi^{-1}TN)$. (i) For $X \in \mathfrak{X}(M)$ and for $\phi_* : T_xM \to T_{\phi(x)}N$ the differentiation of $\phi : M \to N$, let

$$(\phi_* X)(x) := \phi_* X_x \in T_{\phi(x)}N, \qquad x \in M.$$

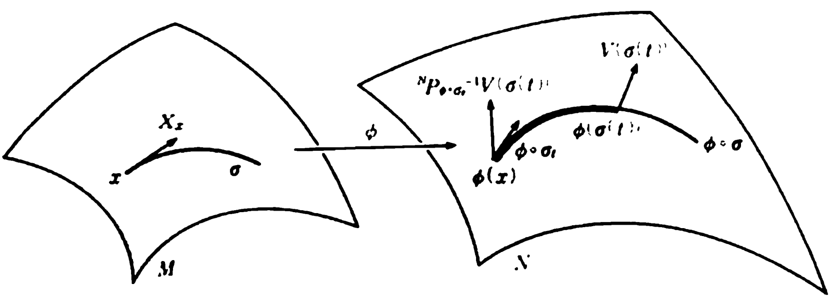



FIGURE 4.2

Then $\phi_* X \in \Gamma(\phi^{-1}TN)$. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame field on an open neighborhood $U$ in $M$, then $\phi_* e_i$, $1 \le i \le m$, are $C^\infty$-sections of $E = \phi^{-1}TN$ over $U$.

(ii) For $Y \in \mathfrak{X}(N)$, let

$$(\phi^{-1}Y)(x) := Y_{\phi(x)} \in T_{\phi(x)}N, \qquad x \in M,$$

then $\phi^{-1}Y \in \Gamma(\phi^{-1}TN)$, denoted also simply by $Y$.

**1.4. The first variation formula.** Now we calculate the first variation of $E$. We take a smooth variation of $\phi$, $\phi_t \in C^\infty(M, N)$, $-\epsilon < t < \epsilon$, with a varition vector field $V \in \Gamma(\phi^{-1}TN)$. That is, $F : (-\epsilon, \epsilon) \times M \to N$, a $C^\infty$-mapping satisfying

$$\begin{cases} F(0, x) = \phi(x), & x \in M, \\ F(t, x) = \phi_t(x), & -\epsilon < t < \epsilon, \; x \in M. \end{cases}$$

We extend vector fields $\frac{\partial}{\partial t}$, $X$ on $(-\epsilon, \epsilon)$, $M$, respectively, to the direct product $(-\epsilon, \epsilon) \times M$ canonically, and we denote them by

$$\left(\frac{\partial}{\partial t}\right)_{(t,x)}, \qquad X_{(t,x)} \quad \text{for} \quad (t, x) \in (-\epsilon, \epsilon) \times M.$$

We also extend $\{e_i\}_{i=1}^m$ to $(-\epsilon, \epsilon) \times M$, denoted by $e_{i(t,x)}$. By definition $V(x) = \frac{d}{dt}\big|_{t=0}\phi_t(x)$, for $x \in M$ , and hence, we get

$$F_*\left(\frac{\partial}{\partial t}\right)_{(0,x)} = V(x), \qquad x \in M,$$

where $F_*$ is the differentiation of $F$.

Now differentiate the function $t \mapsto E(\phi_t)$ at each $t$, and at the final step we set $t = 0$.

$$\frac{d}{dt}E(\phi_t) = \frac{1}{2}\int_M \sum_{i=1}^m \frac{d}{dt}h(\phi_{t*}e_i, \phi_{t*}e_i)\, v_g.$$

Here at each point $x \in M$, we calculate

$$\begin{aligned} \frac{d}{dt}h(\phi_{t*}e_i, \phi_{t*}e_i) &= \frac{d}{dt}h_{\phi_t(x)}(\phi_{t*}e_{ix}, \phi_{t*}e_{ix}) \qquad (\text{ for } \phi_t : M \to N) \\ &= \frac{d}{dt}h_{F(t,x)}(F_*e_{i(t,x)}, F_*e_{i(t,x)}) \\ &\qquad\qquad (\text{ for } F : (-\epsilon, \epsilon) \times M \to N) \\ &= \left(\frac{\partial}{\partial t}\right)_{(t,x)} h(F_*e_i, F_*e_i) \\ &\qquad\qquad \left(\frac{\partial}{\partial t} \text{ is regarded as a vector field on } (-\epsilon, \epsilon) \times M\right) \\ &= 2\, h(\tilde{\nabla}_{\frac{\partial}{\partial t}} F_*e_i, F_*e_i), \end{aligned}$$

in the last two equations, the $F_*e_i$ are regarded as sections of the induced bundle $F^{-1}TN$ over $(-\epsilon, \epsilon) \times M$ by the mapping $F : (-\epsilon, \epsilon) \times M \to N$ and the last eqution follows from the compatibility of $\tilde{\nabla}$ and $h$ of $F^{-1}TN$.

Here we need the following lemma:

LEMMA (1.16). *For any $C^\infty$-mapping $\phi : M \to N$ and $X, Y \in \mathfrak{X}(M)$, we have that*

$$\tilde{\nabla}_X(\phi_* Y) - \tilde{\nabla}_Y(\phi_* X) - \phi_*([X, Y]) = 0.$$

PROOF. We denote the left hand side of the above equality by $\tilde{T}(X, Y)$. Since $\phi_*(fX) = f\phi_* X$, $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, it follows that

$$\tilde{T}(f_1 X, f_2 Y) = f_1 f_2 \tilde{T}(X, Y), \qquad f_1, f_2 \in C^\infty(M), \; X, Y \in \mathfrak{X}(M).$$

So it suffices to show that taking local coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$ of $M$, $N$ at $x_0$, $\phi(x_0)$, respectively,

$$\tilde{T}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0 \text{ at } x_0.$$

But since $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, we only have to show

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}}\left(\phi_* \frac{\partial}{\partial x_j}\right) = \tilde{\nabla}_{\frac{\partial}{\partial x_j}}\left(\phi_* \frac{\partial}{\partial x_i}\right).$$

If we put $\phi^\alpha := y_\alpha \circ \phi$ and $\phi_j^\alpha := \frac{\partial \phi^\alpha}{\partial x_j}$, then

$$\phi_*\left(\frac{\partial}{\partial x_j}\right)_x = \sum_{\alpha=1}^n \phi_j^\alpha(x)\left(\frac{\partial}{\partial y_\alpha}\right)_{\alpha(x)}$$

at each point $x$ in a neighborhood of $x_0$. Now for calculating the above, let $m$ $C^1$-curves $t \mapsto \sigma_i(t)$, $1 \le i \le m$, be defined as

$$\sigma_i(0) = x_0, \qquad \sigma_i'(0) = \left(\frac{\partial}{\partial x_i}\right)_{x_0},$$

then by the definition of $\tilde{\nabla}$, for $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$,

$$\begin{aligned}
(\tilde{\nabla}_X(\phi_* Y))(x_0) &= \frac{d}{dt}\bigg|_{t=0} {}^N P_{\phi\circ(\sigma_i)_t}{}^{-1}\left\{\sum_{\alpha=1}^n \phi_j^\alpha(\sigma_i(t))\left(\frac{\partial}{\partial y_\alpha}\right)_{\phi(\sigma_i(t))}\right\} \\
&= \sum_{\alpha=1}^n \frac{d}{dt}\bigg|_{t=0}\left\{\phi_j^\alpha(\sigma_i(t))\, {}^N P_{\phi\circ(\sigma_i)_t}{}^{-1}\left(\frac{\partial}{\partial y_\alpha}\right)_{\phi(\sigma_i(t))}\right\} \\
&= \sum_{\alpha=1}^n \frac{\partial^2 \phi^\alpha}{\partial x_i \partial x_j}(x_0)\left(\frac{\partial}{\partial y_\alpha}\right)_{\phi(x_0)} + \sum_{\alpha=1}^n \phi_j^\alpha(x_0)\left(\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y_\alpha}\right)(x_0) \\
&= \sum_{\alpha=1}^n \frac{\partial^2 \phi^\alpha}{\partial x_i \partial x_j}(x_0)\left(\frac{\partial}{\partial y_\alpha}\right)_{\phi(x_0)} + \sum_{\alpha, \beta=1}^n \phi_j^\alpha(x_0)\phi_i^\beta(x_0)\left({}^N\nabla_{\frac{\partial}{\partial y_\beta}} \frac{\partial}{\partial y_\alpha}\right)_{\phi(x_0)}.
\end{aligned}$$

By a similar argument, we get

$$(\tilde{\nabla}_Y(\phi_* X))(x_0) = \sum_{\alpha=1}^{n} \frac{\partial^2 \phi^\alpha}{\partial x_j \partial x_i}(x_0)\left(\frac{\partial}{\partial y_\alpha}\right)_{\phi(x_0)} + \sum_{\alpha,\beta=1}^{n} \phi_i^\alpha(x_0)\phi_j^\beta(x_0)\left({}^N\nabla_{\frac{\partial}{\partial y_\beta}}\frac{\partial}{\partial y_\alpha}\right)_{\phi(x_0)}.$$

Here since

$$\frac{\partial^2 \phi^\alpha}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 \phi^\alpha}{\partial x_j \partial x_i}(x_0), \qquad {}^N\nabla_{\frac{\partial}{\partial y_\alpha}}\frac{\partial}{\partial y_\beta} = {}^N\nabla_{\frac{\partial}{\partial y_\beta}}\frac{\partial}{\partial y_\alpha},$$

we obtain the desired result. □

(1.17) *Calculus of the first variation.* Applying Lemma (1.16) to the $C^\infty$-mapping $F : (-\epsilon, \epsilon) \times M \to N$ and $X, Y \in \mathfrak{X}((-\epsilon, \epsilon) \times M)$, we get

$$\tilde{\nabla}_X(F_* Y) - \tilde{\nabla}_Y(F_* Y) - F_*([X, Y]) = 0.$$

Apply this to $X = \frac{\partial}{\partial t}$, $Y = e_i$, since $[\frac{\partial}{\partial t}, e_i] = 0$, we get

$$\begin{aligned} 2h(\tilde{\nabla}_{\frac{\partial}{\partial t}} F_* e_i, F_* e_i) &= 2h(\tilde{\nabla}_{e_i} F_* \frac{\partial}{\partial t}, F_* e_i) \\ &= 2\left\{e_i \cdot h(F_* \frac{\partial}{\partial t}, F_* e_i) - h(F_* \frac{\partial}{\partial t}, \tilde{\nabla}_{e_i} F_* e_i)\right\}, \end{aligned}$$

in the last part of which we use the compatibility of $\tilde{\nabla}$ and $h$. Let $X_t \in \mathfrak{X}(M)$ be determined by

$$g(X_t, Y) = h(F_* \frac{\partial}{\partial t}, F_* Y), \quad \text{for all } Y \in \mathfrak{X}(M),$$

then we get

$$\begin{aligned} \sum_{i=1}^{m} e_i \cdot h\left(F_* \frac{\partial}{\partial t}, F_* e_i\right) &= \sum_{i=1}^{m} e_i \cdot g(X_t, e_i) \\ &= \sum_{i=1}^{m} \{g(\nabla_{e_i} X_t, e_i) + g(X_t, \nabla_{e_i} e_i)\} \quad \text{(by (3.5) in Capter 2)} \\ &= \operatorname{div}(X_t) + \sum_{i=1}^{m} g(X_t, \nabla_{e_i} e_i) \quad \text{(by definition of } \operatorname{div}(X_t)) \\ &= \operatorname{div}(X_t) + \sum_{i=1}^{m} h(F_* \frac{\partial}{\partial t}, F_*(\nabla_{e_i} e_i)) \quad \text{(by definition of } X_t). \end{aligned}$$

Thus, we obtain

$$\frac{d}{dt} e(\phi_t) = \int_M \operatorname{div}(X_t)\, v_g - \int_M h\left(F_* \frac{\partial}{\partial t}, \sum_{i=1}^{m} \{\tilde{\nabla}_{e_i} F_* e_i - F_*(\nabla_{e_i} e_i)\}\right) v_g, \tag{1.18}$$

the first term on the right-hand side of which vanishes because of Green's formula (cf. (3.29) in Chapter 2).

Here putting $t = 0$, since

$$F_*\frac{\partial}{\partial t}(0, x) = V(x), \quad F_*e_i(0, x) = \phi_*e_i(x), \quad F_*\nabla_{e_i}e_i(0, x) = \phi_*\nabla_{e_i}e_i(x),$$

we obtain

$$\frac{d}{dt}\bigg|_{t=0} E(\phi_t) = -\int_M h\left(V, \sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\phi_*e_i - \phi_*\nabla_{e_i}e_i)\right) v_g. \tag{1.19}$$

In the right-hand side of (1.19),

$$\tau(\phi)(x) := \sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\phi_*e_i - \phi_*\nabla_{e_i}e_i)(x), \qquad x \in M \tag{1.20}$$

does not depend on a choice of an orthonormal frame field $\{e_i\}_{i=1}^m$ and defines an element of $\Gamma(\phi^{-1}TN)$ called the **tension field** of $\phi$.

Therefore, we obtain:

THEOREM (1.21) (THE FIRST VARIATION FORMULA). *Let $\phi \in C^\infty(M, N)$. For any smooth variation $\phi_t$, $-\epsilon < t < \epsilon$, of $\phi$, let $V(x) := \frac{d}{dt}\big|_{t=0} \phi_t(x)$, $x \in M$, then*

$$\frac{d}{dt}\bigg|_{t=0} E(\phi_t) = -\int_M h(V, \tau(\phi))\, v_g. \tag{1.22}$$

*Thus, $\phi \in C^\infty(M, N)$ is a harmonic mapping if and only if*

$$\tau(\phi) = 0 \quad \textit{everywhere on } M. \tag{1.23}$$

(1.23) *is called the* **Euler-Lagrange equation.**

The tension field $\tau(\phi)$ can be expressed using the local coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$ in $M$, $N$ as follows. Let $\phi^\alpha = y_\alpha \circ \phi$, and write $\tau(\phi)(x) \in T_{\phi(x)}N$ as

$$\tau(\phi)(x) = \sum_{\gamma=1}^{n} \tau(\phi)^\gamma \frac{\partial}{\partial y_\gamma},$$

$$\begin{aligned}\tau(\phi)^\gamma(x) = \sum_{i,j=1}^{m} g^{ij}\bigg\{&\frac{\partial^2\phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial \phi^\gamma}{\partial x_k} \\ &+ \sum_{\alpha,\beta=1}^{n} {}^N\Gamma_{\alpha\beta}^\gamma(\phi(x))\frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi^\beta}{\partial x_j}\bigg\} \\ = -\Delta\phi^\gamma + \sum_{i,j,\alpha,\beta} &g^{ij\,N}\Gamma_{\alpha\beta}^\gamma(\phi(x))\frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi^\beta}{\partial x_j}.\end{aligned} \tag{1.24}$$

Here $\Gamma_{ij}^k$, ${}^N\Gamma_{\alpha\beta}^\gamma$ are Cristoffel's symbols on $(M, g)$, $(N, h)$, respectively. $\Delta\phi^\gamma$ is defined by the action of the Laplacian (cf. subsection 3.7 in Chapter 2) on $C^\infty$-functions $\phi^\gamma$ on a neighborhood in $M$. By this expression, the Euler-Lagrange equation is a nonlinear equation with $n$ unknown functions $\phi^1, \dots, \phi^n$. But this expression here has a difficulty in that each $\phi_i$ is defined only on a neighborhood in $M$.

## §2. An alternative expression for the first variation

In this section, $N$ is regarded as a closed submanifold of $\mathbf{R}^K$, and $h = \iota^* g_0$ is the pull back of the standard Riemannian metric $g_0$ of $\mathbf{R}^K$ by the inclusion $\iota : N \subset \mathbf{R}^K$. Then $\phi \in C^\infty(M, N)$ can be written as

$$\phi(x) = (\phi_1(x), \dots, \phi_K(x)), \qquad x \in M.$$

Then we shall express the Euler-Lagrange equation in terms of $\phi_A \in C^\infty(M)$, $1 \le A \le K$.

**2.1. Geometry of submanifolds.** Here we prepare briefly basic materials about a closed isometrically embedded submanifold $(N, h)$ of $(\mathbf{R}^K, g_0)$.

We assume that $N$ is a closed submanifold of $\mathbf{R}^K$ and that $h = \iota^* g_0$, where $\iota : N \subset \mathbf{R}^K$ is the inclusion. We denote the differentiation of $\iota : N \subset \mathbf{R}^K$ by $d\iota = \iota_* : T_yN \ni v \mapsto d\iota(v) \in T_y\mathbf{R}^K$. Setting $d\iota(T_yN) = N_y$, we see that $N_y$ can be regarded as a subspace of $\mathbf{R}^K$ under the identification $T_y\mathbf{R}^K \cong \mathbf{R}^K$. Letting $N_y^\perp$ be the orthogonal complement of $N_y$ in $\mathbf{R}^K$ with respect to $\langle\ ,\ \rangle$, we get the orthogonal decomposition:

$$\mathbf{R}^K = N_y \oplus N_y^\perp. \tag{2.1}$$

Any $C^\infty$-vector field on $N$, $W \in \mathfrak{X}(N)$, can be regarded as a $C^\infty$-mapping $W : N \to \mathbf{R}^K$ satisfying

$$W(y) \in N_y \subset \mathbf{R}^K, \qquad y \in N,$$

if we put $W(y) := d\iota(W_y) \in N_y \subset \mathbf{R}^K$:

$$\mathfrak{X}(N) = \{W \in C^\infty(N, \mathbf{R}^K);\ W(y) \in N_y,\ y \in N\}. \tag{2.2}$$

Denoting $W \in \mathfrak{X}(N)$ by $W = (W_1, \dots, W_K)$, all $W_A$, $1 \le A \le K$ are $C^\infty$-functions on $N$, and the $dW_A$ are 1-forms on $N$. Put

$$dW := (dW_1, \dots, dW_K)$$

which is $K$-tuple of 1-forms on $N$. For each tangent vector $u \in T_yN$ of $N$ at $y \in N$, $dW_A(u) = u \cdot W_A \in \mathbf{R}$, $1 \le A \le K$, which is the differentiation of $W_A$ with respect to the direction $u$, and we set

$$dW(u) := (dW_1(u), \dots, dW_K(u)) \in \mathbf{R}^K. \tag{2.3}$$

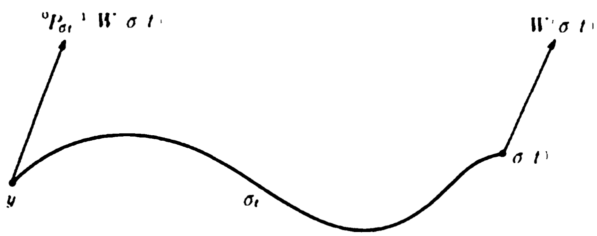



FIGURE 4.3

Let $\nabla^0$ be the Levi-Civita connection on the Euclidean space $(\mathbf{R}^K, g_0)$. Then

$$(\nabla^0{}_u W)_y = u \cdot W = dW(u), \qquad u \in T_y N,\ y \in N. \tag{2.4}$$

PROOF. Let $\sigma(t)$ be any $C^1$-curve in $N$ satisfying that $\sigma(0) = y$, $\sigma'(0) = u$. Regarding $\sigma(t)$ as a curve in $\mathbf{R}^K$, we have

$$(\nabla^0{}_u W)_y = \left.\frac{d}{dt}\right|_{t=0} {}^0P_{\sigma_t}{}^{-1} W(\sigma(t)),$$

where ${}^0P_{\sigma_t}$ is the parallel transport of $\nabla^0$ in $\mathbf{R}^K$ along a curve $\sigma_t$ being $\sigma_t(s) = \sigma(s)$, $0 \leq s \leq t$. That is, ${}^0P_{\sigma_t}$ sends any vector at $y$ to the one at $\sigma(t)$, being parallel in $\mathbf{R}^K$ as in Figure 4.3. Thus the right-hand side coincides with

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} W(\sigma(t)) &= \left(\left.\frac{d}{dt}\right|_{t=0} W_1(\sigma(t)), \cdots, \left.\frac{d}{dt}\right|_{t=0} W_K(\sigma(t))\right) \\ &= (dW_1(u), \cdots, dW_K(u)) = dW(u). \quad \square\end{aligned}$$

Now decompose $dW(u) \in \mathbf{R}^K$ according to the orthogonal decomposition (2.1) $\mathbf{R}^K = N_y \oplus N_y{}^\perp$,

$$dW(u) = (\nabla'{}_u W)_y + A_y(u, W),$$

where $(\nabla'_u W)_y \in N_y$, and $A_y(u, W) \in N_y{}^\perp$.

Then for $f \in C^\infty(M)$, $W \in \mathfrak{X}(N)$, and $u \in T_y N$ we get:

(2.5) $\nabla'_u(fW) = df(u)\, W + f(y)\, \nabla'_u W$.

(2.6) $A_y(u, fW) = f(y)\, A_y(u, W)$.

PROOF. Since $d(fW_A)(u) = df(u) W_A + f\, dW_A(u)$, $1 \leq A \leq K$,

$$d(fW)(u) = df(u)\, W + f\, dW(u).$$

Comparing the $N_y$- $N_y{}^\perp$-components of this, respectively, we get (2.5), (2.6). $\square$

Note that $\nabla'_u W$ and $A_y(u, W)$ are linear in $W$ since $dW(u)$ is linear in $W$.

(2.6) implies that $A_y(u, W)$ depends only on $W_y \in T_yN$, and we get a bilinear mapping

$$A_y : T_yN \times T_yN \to N_y{}^{\perp} \subset \mathbb{R}^K.$$

We call $A$ the **second fundamental form** of the submanifold $\iota : N \subset \mathbb{R}^K$.

Moreover, (2.5) implies that if we define for $Z$, $W \in \mathfrak{X}(N)$, and $y \in N$,

$$(\nabla'_Z W)_y := \nabla'_{Z(y)} W,$$

then $\nabla'$ gives a connection on $N$, (cf. (3.4) in Chapter 2).

Now we show that for $Z$, $W \in \mathfrak{X}(N)$, $y \in N$, we have

$$[Z, W]_y = dW(Z(y)) - dZ(W(y)). \tag{2.7}$$

Proof. Extend $Z$, $W$ to vector fields $Z'$, $W'$ on a neighborhood $U$ in $\mathbb{R}^K$ of $y \in N$ which coincide with $Z$, $W$ on $U \cap N$, respectively. Then the right hand side is equal to

$$\begin{aligned}[Z', W']_y &= (\nabla^0_{Z'} W')_y - (\nabla^0_{W'} Z')_y \\ &= dW'(Z'(y)) - dZ'(W'(y))\end{aligned}$$

which is equal to the right-hand side of (2.7). □

In particular, compare both sides of the $N_y$-, and the $N_y{}^{\perp}$-components of (2.7), we obtain for all $Z$, $W \in \mathfrak{X}(N)$, $y \in N$, that

$$[Z, W]_y = \nabla'_{Z(y)} W - \nabla'_{W(y)} Z, \tag{2.8}$$

$$A_y(Z(y), W) = A_y(W(y), Z). \tag{2.9}$$

Let $(u_1, \dots, u_K)$ be the standard coordinate of $\mathbb{R}^K$. Then

$$g_0 = \sum_{A=1}^{K} du_A \otimes du_A.$$

For $Z \in \mathfrak{X}(N)$, we make the identification

$$d\iota(Z_y) = Z(y) = (Z_1(y), \dots, Z_K(y)) = \sum_{A=1}^{N} Z_A(y)\left(\frac{\partial}{\partial u_A}\right)_y.$$

Then by $h = \iota^* g_0$, we obtain

$$h(Z, W) = \sum_{A=1}^{N} Z_A W_A, \qquad Z, W \in \mathfrak{X}(N)$$

which is also denoted by $(Z, W)$.

Then we obtain for $u \in T_yN$, $Z$, $W \in \mathfrak{X}(N)$,

$$u \cdot h(Z, W) = h(\nabla'_u Z, W) + h(Z, \nabla'_u W). \tag{2.10}$$

**Proof.** Let $\sigma(t)$ be a $C^1$-curve in $N$ satisfying $\sigma(0) = y$, $\sigma'(0) = u$, then

$$\begin{aligned} u \cdot h(Z, W) &= \frac{d}{dt}\Big|_{t=0} \sum_{A=1}^{K} Z_A(\sigma(t))\, W_A(\sigma(t)) \\ &= \langle dZ(u), W\rangle + \langle Z, dW(u)\rangle \\ &= \langle \nabla^0_u Z, W\rangle + \langle Z, \nabla^0_u W\rangle \\ &= h(\nabla'_u Z, W) + h(Z, \nabla'_u W) \end{aligned}$$

which implies (2.10). □

Thus, by (2.8), (2.10), and Theorem (3.5) in Chapter 2, $\nabla'$ coincides with the Levi-Civita connection ${}^N\nabla$ on $(N, h)$.

Summing up the above, we obtain

**Proposition (2.11).** (i) *The second fundamental form* $A_y : T_yN \times T_yN \to N_y^\perp \subset \mathbf{R}^K$ *is a symmetric bilinear form, and for all* $Z$, $W \in \mathfrak{X}(N)$, $N \ni y \mapsto A_y(Z(y), W(y))$ *is* $C^\infty$.

(ii) *The connection* $\nabla'$ *is the Levi-Civita connection* ${}^N\nabla$ *of* $(N, h)$.

**Proposition (2.12).** *Assume that a vector field* $\xi$ *of* $\mathbf{R}^K$ *defined only on a neighborhood* $U$ *in* $N$ *satisfies*

$$\xi(y) \in N_y{}^\perp, \qquad y \in U.$$

*According to the decomposition* $\mathbf{R}^K = N_y \oplus N_y{}^\perp$, *write*

$$(\nabla^0_Z \xi)_y = -(A_\xi Z)_y + (D_Z \xi)_y. \tag{2.13}$$

*Then we obtain*

$$\langle A_\xi Z, W\rangle = \langle A(Z, W), \xi\rangle, \qquad Z, W \in \mathfrak{X}(N). \tag{2.14}$$

**Proof.** By the assumption, for $Z$, $W \in \mathfrak{X}(N)$, we get $\langle W, \xi\rangle = 0$. Thus, we get

$$\begin{aligned} 0 &= \langle \nabla^0_Z W, \xi\rangle + \langle W, \nabla^0_Z \xi\rangle \\ &= \langle {}^N\nabla_Z W + A(Z, W), \xi\rangle + \langle W, -A_\xi Z + D_Z \xi\rangle \\ &= \langle A(Z, W), \xi\rangle - \langle W, A_\xi Z\rangle. \end{aligned}$$

Thus, we obtain (2.14). □

Now for $X$, $Y$, $Z \in \mathfrak{X}(N)$, let $X(Y(Z))$ be the twice differentiation of a $C^\infty$-function $Z : N \to \mathbf{R}^K$ first in the $Y$-direction, and then in the $X$-direction. Then by the definition of $A$, (2.11), and (2.12), we get

$$\begin{aligned} X(Y(Z)) &= X({}^N\nabla_Y Z + A(Y, Z)) \\ &= {}^N\nabla_X({}^N\nabla_Y Z) + A(X, {}^N\nabla_Y Z) - A_{A(Y,Z)}X + D_X(A(Y, Z)), \end{aligned}$$

changing $X$, $Y$ mutually, we get also

$$Y(X(Z)) = {}^N\nabla_Y({}^N\nabla_X Z) + A(Y, {}^N\nabla_X Z) - A_{A(X,Z)}Y + D_Y(A(X, Z)).$$

And we get

$$[X, Y](Z) = {}^N\nabla_{[X,Y]}Z + A([X, Y], Z).$$

Thus, computing the $N_y$-component of

$$0 = X(Y(Z)) - Y(X(Z)) - [X, Y](Z),$$

we obtain

$$0 = {}^N R(X, Y)Z - A_{A(Y,Z)}X + A_{A(X,Z)}Y.$$

Together with (2.14), we obtain

$$h({}^N R(X, Y)Z, W) = (A(Y, Z), A(X, W)) - (A(X, Z), A(Y, W)).$$

Summing up the above, we obtain

**Theorem** (2.15) (Gauss). *For any closed submanifold $(N, h)$ of $(\mathbf{R}^K, g_0)$ satisfying $h = \iota^* g_0$, with the inclusion $\iota : N \subset \mathbf{R}^K$, the curvature tensor ${}^N R$ satisfies*

$$h({}^N R(X, Y)Z, W) = (A(Y, Z), A(X, W)) - (A(X, Z), A(Y, W)),$$

*for $X, Y, Z \in \mathfrak{X}(N)$.*

In particular, for the case of the unit sphere $N = S^n = \{y \in \mathbf{R}^K;\ (y, y) = 1\} \subset \mathbf{R}^K$ with $K = n + 1$, the second fundamental form $A$ and the curvature tensor $R$ are given as follows.

Let $\xi$ be the $C^\infty$-vector field defined on $S^n$ which is the unit normal outerward to $S^n$, i.e., $\xi(y) \in T_y S^{n\perp}$, $y \in S^n$. Then we can identify $\xi(y) = y$, with $y \in S^n$. Then we obtain

$$A(Z, W) = -(Z, W)\xi. \tag{2.16}$$

Thus, we obtain by Theorem (2.15),

$$g_{S^n}(R(X, Y)Z, W) = g_{S^n}(Y, Z)\, g_{S^n}(X, W) - g_{S^n}(X, Z)\, g_{S^n}(Y, W), \tag{2.17}$$

for $X, Y, Z, W \in \mathfrak{X}(S^n)$. Thus, the sectional curvature of $(S^n, g_{S^n})$ is one everywhere.

**Proof of** (2.16). We apply Theorem (2.15) to $(N, h) = (S^n, g_{S^n})$. It suffices to show

$$h(A(Z, W), \xi) = -h(Z, W), \qquad Z, W \in \mathfrak{X}(N).$$

We calculate

$$\begin{aligned} h(A(Z, W), \xi) &= (dW(Z), \xi) \\ &= \sum_{A=1}^{n+1} dW_A(Z)\xi_A \\ &= d\left(\sum_{A=1}^{n+1} W_A \xi_A\right)(Z) - \sum_{A=1}^{n+1} W_A\, d\xi_A(Z). \end{aligned}$$

Here since $\langle W, \xi \rangle = 0$ everywere, so the first term vanishes. For the second term, since $\xi(y) = y$, we have $\xi_A = u_A$, $1 \le A \le n+1$. Since $Z = \sum_{A=1}^{n+1} Z_A \frac{\partial}{\partial u_A}$, we get $d\xi_A(Z) = du_A(Z) = Z_A$. Thus,

$$\sum_{A=1}^{n+1} W_A d\xi_A(Z) = \sum_{A=1}^{n+1} W_A Z_A = h(W, Z)$$

which implies (2.16). □

**2.2. The first variation formula.** We calculate the first variation formula of the energy in an alternative way. Recall that for any $\phi \in C^\infty(M, N)$, the space of variation vector fields along $\phi$ is

$$\begin{aligned} T_\phi C^\infty(M, N) &= \Gamma(\phi^{-1}TN) \\ &= \{W : M \to TN,\ C^\infty\text{-mapping},\ V(x) \in T_{\phi(x)}N,\ x \in M\} \\ &= \{W : M \to \mathbf{R}^K,\ C^\infty\text{-mapping},\ V(x) \in N_{\phi(x)},\ x \in M\}, \end{aligned}$$

identifying $T_yN \cong N_y \subset \mathbf{R}^K$, $y \in N$. Then we get

LEMMA (2.18). *For $\phi \in C^\infty(M, N)$, $W \in \Gamma(\phi^{-1}TN)$, $u \in T_xM$, $x \in M$, let*

$$dW(u) := (dW_1(u), \dots, dW_K(u)),$$

*where $dW_A$, $1 \le A \le K$, are 1-forms on $M$. Then the $N_{\phi(x)}{}^\perp$-component of $dW(u)$ coincides with*

$$A_{\phi(x)}(d\phi(u), W(x)), \qquad u \in T_xM,$$

*where $d\phi$ is the differentiation of $\phi$.*

PROOF. Let $Y_1, \dots, Y_n$ be $C^\infty$-vector fields on a neighborhood $V$ in $N$ satisfying $\{Y_1(y), \dots, Y_n(y)\}$ is a basis of $N_y$ at each point $y \in N$. Then since $W(z) \in N_{\phi(z)}$, $z \in \phi^{-1}(V)$, it can be written as

$$W(z) = \sum_{\alpha=1}^{n} \xi_\alpha(z)\, Y_\alpha(\phi(z)).$$

Since $\xi_\alpha \in C^\infty(\phi^{-1}(V))$,

$$dW(u) = \sum_{\alpha=1}^{n} \{(u \cdot \xi_\alpha)\, Y_\alpha(\phi(x)) + \xi_\alpha(x)\, dY_\alpha(d\phi(u))\}.$$

Here since $Y(\phi(x)) \in N_{\phi(x)}$, and by definition of $A$, the $N_{\phi(x)}{}^\perp$-component of $dW(u)$ conicides with

$$\begin{aligned} N_{\phi(x)}{}^{\perp}\text{-component of } &\sum_{\alpha=1}^{n} \xi_\alpha(x)\, dY_\alpha(d\phi(x)) \\ &= A_{\phi(x)}\left(d\phi(u), \sum_{\alpha=1}^{n} \xi_\alpha(x)\, Y_\alpha(\phi(x))\right) \\ &= A_{\phi(x)}(d\phi(u), W(x)) \end{aligned}$$

which implies Lemma (2.18). □

Now take any smooth variation of $\phi \in C^\infty(M, N)$, $\phi_t$, $-\epsilon < t < \epsilon$, with $\phi_0 = \phi$. Using the inclusion $\iota : N \subset \mathbb{R}^K$, we can write

$$\phi_t(x) = (\phi_{t1}(x), \dots, \phi_{tK}(x)), \qquad x \in M,$$

then the variation vector field $W(x) = \frac{d}{dt}\big|_{t=0}\phi_t(x)$, $x \in M$, can be written as

$$W(x) = (W_1(x), \dots, W_K(x)), \qquad W_A(x) = \frac{d}{dt}\bigg|_{t=0} \phi_{tA}(x), \qquad 1 \le A \le K.$$

The constraint condition $\phi_t(M) \subset N$ yields that

$$W(x) \in N_{\phi(x)}, \qquad x \in M.$$

Furthermore, we obtain by (1.5),

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} E(\phi_t) &= \int_M \frac{d}{dt}\bigg|_{t=0} e(\phi_t)\, v_g \\ &= \frac{1}{2}\sum_{A=1}^{K} \int_M \frac{d}{dt}\bigg|_{t=0} (d\phi_{tA}, d\phi_{tA})\, v_g \\ &= \sum_{A=1}^{K} \int_M (dW_A, d\phi_A)\, v_g \\ &= \sum_{A=1}^{K} \int_M (W_A, \Delta\phi_A)\, v_g = \int_M (W, \Delta\phi)\, v_g \end{aligned}$$

from (3.29), (3.27) in Chapter 2. Here we denote $\Delta\phi := (\Delta\phi_1, \dots, \Delta\phi_K)$. Since $W \in C^\infty(M, \mathbb{R}^K)$ is any element satisfying $W(x) \in N_{\phi(x)}$, for $x \in M$, we obtain that

$$\phi \text{ is harmonic} \iff \text{the } N_{\phi(x)}\text{-component of } \Delta\phi(x) = 0, \qquad x \in M. \tag{2.19}$$

But we shall show that the $N_{\phi(x)}{}^{\perp}$-component of $\Delta\phi$ coincides with

$$-\sum_{i=1}^{m} A_{\phi(x)}(d\phi(e_i), d\phi(e_i)), \tag{2.20}$$

where $A$ is the second fundamental form of $N \subset \mathbb{R}^K$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame field in a neighborhood $U$ of $x$ in $M$.

Indeed, since (3.28) in Chapter 2, we get

$$\Delta\phi_A = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial \phi_A}{\partial x_j}\right),$$

where $g$ is the determinant of a matrix $(g_{ij}) = \left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)$. Then define a $C^\infty$-mapping $X_i : U \to \mathbb{R}^K$, $1 \le i \le m$, by

$$X_i := \sum_{i=1}^m \sqrt{g}\, g^{ij}\, d\phi\left(\frac{\partial}{\partial x_j}\right). \tag{2.21}$$

That is, if we set $X_i = (X_{i1}, \dots, X_{iK})$, then we get

$$X_{iA} = \sum_{j=1}^m \sqrt{g}\, g^{ij}\, d\phi_A\left(\frac{\partial}{\partial x_j}\right) = \sum_{j=1}^m \sqrt{g}\, g^{ij} \frac{\partial \phi_A}{\partial x_j}.$$

Since $d\phi\left(\frac{\partial}{\partial x_j}\right) \in N_{\phi(x)}$, $X_i(x) \in N_{\phi(x)}$, $x \in U$ and

$$\Delta\phi = -\frac{1}{\sqrt{g}} \sum_{i=1}^m \frac{\partial}{\partial x_i} X_i = -\frac{1}{\sqrt{g}} \sum_{i=1}^m dX_i\left(\frac{\partial}{\partial x_i}\right).$$

Thus, using Lemma (2.18), we obtain that the ${N_{\phi(x)}}^\perp$-component of $\Delta\phi(x)$ coincides with the ${N_{\phi(x)}}^\perp$-component of $-\frac{1}{\sqrt{g}} \sum_{i=1}^m dX_i\left(\frac{\partial}{\partial x_i}\right)$ which is equal to

$$\begin{aligned} &-\frac{1}{\sqrt{g}} \sum_{i=1}^m A_{\phi(x)}\left(d\phi\left(\frac{\partial}{\partial x_i}\right), X_i(x)\right) \\ &= -\sum_{i,j=1}^m A_{\phi(x)}\left(d\phi\left(\frac{\partial}{\partial x_i}\right), d\phi\left(\frac{\partial}{\partial x_j}\right)\right) g^{ij} \\ &= -\sum_{i=1}^m A_{\phi(x)}(d\phi(e_i), d\phi(e_i)). \end{aligned}$$

Thus, we obtain the following theorem.

**THEOREM** (2.22) (The first variation formula). *A sufficient and necessary condition for $\phi \in C^\infty(M, N)$ to be harmonic is*

$$\Delta\phi(x) + \sum_{i=1}^m A_{\phi(x)}(d\phi(e_i), d\phi(e_i)) = 0, \qquad x \in M, \tag{2.23}$$

*where $A$ is the second fundamental form of $N \subset \mathbb{R}^K$, $\Delta\phi = (\Delta\phi_1, \dots, \Delta\phi_K)$, and $\phi = (\phi_1, \dots, \phi_K)$.*

**REMARK.** For the equation (2.23), the unknown functions $\phi_A$, $1 \le A \le K$ are well defined everywhere on $M$, so it is much easier to treat than the equation $\tau(\phi) = 0$.

In particular, if $(N, h) = (\mathbf{R}^K, g_0)$ with $n = K$, then the second fundamental form should be $A = 0$. If $(N, h) = (S^n, g_{S^n})$, the unit sphere, then by (2.16), for any $C^\infty$-mapping $\phi: (M, g) \to (S^n, g_{S^n})$,

$$\sum_{i=1}^m A_{\phi(x)}(d\phi(e_i), d\phi(e_i)) = -\sum_{i=1}^m (d\phi(e_i), d\phi(e_i))\, \phi(x) = -2e(\phi)\, \phi(x).$$

Thus we obtain:

COROLLARY (2.24). (I) *A necessary and sufficient condition for $\phi: (M, g) \to (\mathbf{R}^n, g_0)$ to be harmonic is*

$$\Delta\phi = 0$$

*which implies $\phi$ is constant when $M$ is compact. Here $\Delta\phi = (\Delta\phi_1, \dots, \Delta\phi_n)$, and $\phi = (\phi_1, \dots, \phi_n)$.*

(II) *A necessary and sufficient condition for $\phi: (M, g) \to (S^n, g_{S^n})$ to be harmonic is*

$$\Delta\phi(x) = 2e(\phi)\, \phi(x), \qquad x \in M, \tag{2.25}$$

*that is*

$$\Delta\phi_A = 2e(\phi)\, \phi_A, \qquad 1 \le A \le n+1, \tag{2.25'}$$

*where $\phi = (\phi_1, \dots, \phi_{n+1})$.*

REMARK. If $\phi$ is an isometric immersion of $(M, g)$ into $(S^n, g_{S^n})$, this Corollary (2.24) is called **T. Takahashi's Theorem** (1966) (see also Chapter 6).

## §3. Examples of harmonic mappings

(3.1) **Example 1** (Constant maps). For two compact Riemannian manifolds $(M, g)$, $(N, h)$ and a fixed $q \in N$, any constant mapping $\phi: M \to N$,

$$\phi(x) = q, \quad \text{for all } x \in M$$

is harmonic. In fact, $\phi$ is constant if and only if the energy density function $e(\phi)$ vanishes which is equivalent to $E(\phi) = 0$. Thus for any smooth varition of $\phi$, $\phi_t$, $-\epsilon < t < \epsilon$, with $\phi_0 = \phi$, we have

$$0 = E(\phi) \le E(\phi_t), \qquad -\epsilon < t < \epsilon,$$

which implies that $\frac{d}{dt}\big|_{t=0} E(\phi_t) = 0$. It is also easy to check $\tau(\phi) = 0$.

(3.2) **Example 2** (Harmonic functions). $(N, g) = (\mathbf{R}^K, g_0)$. Since ${}^N\Gamma^\gamma_{\alpha\beta} = 0$, denoting $\phi = (\phi_1, \dots, \phi_n)$ for $\phi \in C^\infty(M, \mathbf{R}^K)$,

$$\phi: (M, g) \to (\mathbf{R}^K, g_0) \text{ is harmonic} \iff \phi_i \text{ is harmonic}, \quad \text{i.e., } \Delta\phi_i = 0,\ 1 \le i \le n.$$

When $M$ is compact, $\phi$ must be a constant (see Corollary (2.24)).

(3.3) **Example 3** (Geodesics). Let $(N, h)$ be any (compact) $C^\infty$-Riemannian manifold, and $\dim(M) = m = 1$, i.e., $M = \mathbf{R}/\mathbf{Z} = S^1$. Then

$C^\infty(S^1, N)$ is the totality of all periodic $C^\infty$-curves in $N$ with period one. Let $e_1 = \frac{\partial}{\partial x}$, $x \in \mathbf{R}$, then $\nabla_{e_1} e_1 = 0$. Put $\phi' = \phi^* e_1 = d\phi(e_1)$, then for $\phi$ to be harmonic, $\tau(\phi) = 0$, everywhere on $M$, is equivalent to

$$\tilde{\nabla}_{e_1} \phi_* e_1 = 0 \Longleftrightarrow {}^N\nabla_{\phi'} \phi' = 0$$

which is the equation of geodesics (cf. (3.11) of Chapter 2).

On the other hand, assume that $(N, h)$ is isometrically embedded into $(\mathbf{R}^K, g_0)$. Then the condition (2.23) for a $C^\infty$-mapping $\phi : S^1 \to \mathbf{R}^K$ satisfying $\phi(S^1) \subset N$ to be harmonic is

$$-\phi''(x) + A_{\phi(x)}(\phi'(x), \phi'(x)) = 0. \tag{3.4}$$

In the case of the unit sphere $(S^n, g_{S^n})$, by (2.16), (3.4) is

$$\phi''(x) + \langle \phi'(x), \phi'(x) \rangle \phi(x) = 0,$$

which is the equation treated in subsection 3.4 in Chapter 1. As shown there, (3.4) is easier than the eqation ${}^N\nabla_{\phi'}\phi' = 0$ to treat in the case where the embedding $\iota : N \subset \mathbf{R}^K$ is known.

(3.5) **Example 4** (Minimal isometric immersions). A $C^\infty$-mapping $\phi : (M, g) \to (N, h)$ is said to be an **isometric immersion** if

(i) for each $x \in M$, the differentiation $\phi_* : T_x M \to T_{\phi(x)} N$ is injective, and

(ii) $\phi^* h = g$.

(Note that (ii) implies (i). If (i) holds, $g := \phi^* h$ gives a metric on $M$.)

In this case, we shall identify $x \in M$ with $\phi(x) \in N$, and identify $X \in \mathfrak{X}(M)$ with $\phi_* X$.

For each $x \in M$, we decompose

$$T_x N = T_x M \oplus T_x M^\perp,$$

with respect to $g_x$. According to this, decompose ${}^N\nabla_X Y$ as

$${}^N\nabla_X Y = \nabla_X Y + A(X, Y),$$

for $X, Y \in \mathfrak{X}(M)$. Then $A$ induces a symmetric bilinear mapping $T_x M \times T_x M \to T_x N^\perp$, called the **second fundamental form** of the isometric immersion $\phi$ (which is the same as in subsection 2.1). If

$$\operatorname{tr}(A) := \sum_{i=1}^{m} A(e_i, e_i) = 0, \tag{3.6}$$

$\phi$ is called a **minimal isometric immersion.** Here $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of $(M, g)$. It is known (cf. [Wa]) that this condition (3.6) is equivalent to the condition that

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Vol}(M, \phi_t{}^* h) = 0$$

for all smooth variation of immersions $\phi_t$ whose varition vector field $V$ satisfies $V(x) \in T_xM^{\perp}$, $x \in M$. In this case,

$$\sum_{i=1}^{m} A(e_i, e_i) = \sum_{i=1}^{m} ({}^N\nabla_{e_i} e_i - \nabla_{e_i} e_i) = \tau(\phi),$$

so for an isometric immersion $\phi : (M, g) \to (N, h)$, the minimality is equivalent to harmonicity.

(3.7) **Example 5** (Riemannian submersions).

DEFINITION. An onto mapping $\phi: (M, g) \to (N, h)$ is called a **Riemannian submersion** if

(i) for each point $x \in M$, the differentiation $\phi_* : T_xM \to T_{\phi(x)}N$ is surjective, and

(ii) for each point $x \in M$, there exists a unique orthogonal decomposition

$$T_xM = V_x \oplus H_x,$$

with respect to $g_x$, with the property that for each $x \in M$,

$$V_x = \mathrm{Ker}(\phi_*) = \{u \in T_xM;\ \phi_*(u) = 0\},$$

and the restriction of $\phi_*$ to $H_x$, $\phi_*|_{H_x}$ is an isometry of $(H_x, g_x)$ onto $(T_{\phi(x)}N, h_{\phi(x)})$.

The subspaces $V_x$, $H_x$ are called the **vertical one, horizontal one**, respectively. For each $x \in M$, $\phi^{-1}(\phi(x))$ is called the **fiber** through $x$. By definition, $\dim(M) \geq \dim(N)$. See Figure 4.4.

(3.8) Let $\phi: (M, g) \to (N, h)$ be a Riemannian submersion. Then, for each $y \in N$, $\phi^{-1}(y)$ is an $(m-n)$-dimensional closed submanifold of $M$.

PROOF. Let $x \in \phi^{-1}(y)$. Take a coordinate neighborhood $U_\alpha$ of $x$ which can be regarded as $U_\alpha \subset \mathbf{R}^m$. Since $\mathrm{Ker}(\phi_{*x}) \subset T_xM = T_x\mathbf{R}^m \cong \mathbf{R}^m$, we can choose a linear mapping $L: \mathbf{R}^m \to \mathbf{R}^{m-n}$, which is injective on $\mathrm{Ker}\,(\phi_{*x})$, and define a $C^\infty$-mapping

$$\Phi: U_\alpha \ni z \mapsto (\phi(z), L(z)) \in N \times \mathbf{R}^{m-n},$$

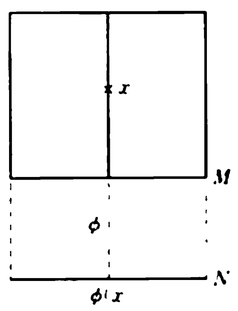



FIGURE 4.4

then the differentiation $\Phi_*$ of $\Phi$ satisfies

$$\Phi_{*x}(v) = (\phi_*(v), L(v)), \qquad v \in T_xM,$$

and $\Phi_{*x}$ is a linear isomorphism by definition of $L$. Thus, by the inverse function theorem (1.30) in Chapter 2, $\Phi$ is a diffeomorphism of some neighborhood $U$ of $x$ in $U_\alpha$ onto a neighborhood $V$ of $(y, L(x))$ in $N \times \mathbf{R}^{m-n}$. Then $\phi^{-1}(y) \cap U$ corresponds to $(\{y\} \times \mathbf{R}^{m-n}) \cap V$ by $\Phi$, and then this gives a coordinate neighborhood of $\phi^{-1}(y)$. □

A $C^\infty$-vector field $X^*$ on $M$ is called a **horizontal lift** of a $C^\infty$-vector field $X$ on $N$ if $\phi_* X^*{}_x = X_{\phi(x)}$ and $X^*{}_x \in H_x$, $x \in M$. By definition, for $X \in \mathfrak{X}(N)$, there exists a unique horizontal lift $X^*$ of $X$.

LEMMA (3.9) (O'Neill's formula). *Let $X, Y \in \mathfrak{X}(N)$. Then it holds that*

(i) $g(X^*{}_x, Y^*{}_x) = h_{\phi(x)}(X_{\phi(x)}, Y_{\phi(x)})$, $x \in M$, *where we denote by* $g(X^*, Y^*) = h(X, Y) \circ \phi$.

(ii) $\phi_*([X^*, Y^*]) = [X, Y]$,

(iii) $\phi_*(\nabla_{X^*} Y^*) = {}^N\nabla_X Y$.

PROOF. (i), (ii) follow from the definition. For (iii), it suffices to show

$$2h(\phi_*(\nabla_{X^*} Y^*), Z) \circ \phi = 2h({}^N\nabla_X Y, Z) \circ \phi, \qquad Z \in \mathfrak{X}(N).$$

We use (3.6) in Chapter 2, (i) and (ii). In fact, the left-hand side of the above coincides with

$$\begin{aligned} 2h(\phi_*\nabla_{X^*} Y^*, \phi_* Z^*) \circ \phi = 2g(\nabla_{X^*} Y^*, Z^*) \\ &= X^* g(Y^*, Z^*) + Y^* g(Z^*, X^*) - Z^* g(X^*, Y^*) \\ &\quad + g(Z^*, [X^*, Y^*]) + g(Y^*[Z^*, X^*]) \\ &\quad - g(X^*, [Y^*, Z^*]), \end{aligned}$$

here we get

$$\begin{aligned} X^* g(Y^*, Z^*) &= X^*(h(Y, Z) \circ \phi) = (X\, h(Y, Z)) \circ \phi, \\ g(X^*, [Y^*, Z^*]) &= h(X, [Y, Z]) \circ \phi, \text{ etc.}, \end{aligned}$$

so the above coincides with $2h({}^N\nabla_X Y, Z) \circ \phi$. □

PROPOSITION (3.10). *A sufficient and necessary condition for a Riemannan submersion $\phi : (M, g) \to (N, h)$ to be a harmonic mapping is that for any $x \in M$, the inclusion $\iota : \phi^{-1}(\phi(x)) \subset M$ is a minimal submanifold of $(M, g)$. Here the Riemannian metric on $\phi^{-1}(\phi(x))$ is taken as $\iota^* g$.*

PROOF. Let $m = \dim(M)$, $n = \dim(N)$, and let $\{e'_1, \ldots, e'_n\}$ be a local orthonormal frame field defined on a neighborhood $V$ in $(N, h)$. Let $\{e_1, \ldots, e_n\}$ be the horizontal lift on $\phi^{-1}(V)$ of $\{e'_1, \ldots, e'_n\}$, and

$\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ an orthonormal frame field defined on a neighborhood $U$ in $\phi^{-1}(V)$. Then at each point $x \in U$, $\{e_{n+1}, \dots, e_m\}$ generates the vertical subspace $V_x$ of $T_xM$. Then we decompose the tension field as

$$\tau(\phi) = \sum_{i=1}^{m}\{\tilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i}e_i\} = \sum_{i=1}^{n}\{''\} + \sum_{i=n+1}^{m}\{''\},$$

where we get $\tilde{\nabla}_{e_i}\phi_* e_i = {}^N\nabla_{e_i'}e_i' = \phi_*\nabla_{e_i}e_i$, $1 \le i \le m$ by (iii) of Lemma (3.9). And we get $\tilde{\nabla}_{e_i}\phi_* e_i = 0$, $n+1 \le i \le m$. Thus, we obtain

$$\tau(\phi) = -\sum_{i=n+1}^{m}\phi_*\nabla_{e_i}e_i = -\phi_*\Big(\sum_{i=n+1}^{m}\nabla_{e_i}e_i\Big).$$

Thus, a necessary and sufficient condition for $\phi : (M, g) \to (N, h)$ to be harmonic is

$$\sum_{i=n+1}^{m}\nabla_{e_i}e_i \in V_x, \qquad x \in M.$$

On the other hand, the trace $\mathrm{tr}(A)$ of the second fundamental form $A$ of the inclusion $\iota : (\phi^{-1}(\phi(x)), \iota^* g) \to (M, g)$, is the $H_x$-component of $\sum_{i=n+1}^{m}\nabla_{e_i}e_i$. Thus, we obtain the desired result. □

COROLLARY (3.11). *For a compact Riemannian manifold* $(M, g)$, *both*
(i) *the identity mapping* id: $(M, g) \to (M, g)$, *and*
(ii) *the Riemannian covering* $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$
*are harmonic mappings.*

(3.12) **Example 6** (Holomorphic mappings). An even dimensional (say $2m$) $C^\infty$-manifold $M$ is called to be an $m$-dimensional **complex manifold** if at each point of $M$ there exists a complex coordinate neighborhood $(U_\alpha, \alpha)$ such that for a homeomorphism $\alpha$ of $U_\alpha$ onto $\alpha(U_\alpha) \subset \mathbf{C}^m$, if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\beta \circ \alpha^{-1} : \mathbf{C}^m \supset \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta) \subset \mathbf{C}^m$$

is a holomorphic diffeomorphism. Taking a complex coordinate $(z_1^\alpha, \dots, z_m^\alpha) \in \mathbf{C}^m$ on a neighborhood $U_\alpha$, put

$$z_j^\alpha = x_j^\alpha + \sqrt{-1}\,y_j^\alpha, \qquad x_j^\alpha, y_j^\alpha \in \mathbf{R}, \qquad j = 1, \dots, m,$$

and then $(x_1^\alpha, y_1^\alpha, \dots, x_m^\alpha, y_m^\alpha)$ gives a local coordinate as a $2m$-dimensional real manifold.

Complexify the tangent space $T_pM$ at each point $p \in M$, $T_p^{\mathbf{C}}M = T_pM \otimes \mathbf{C}$, and put

$$T^{\mathbf{C}}M = \bigcup_{p \in M} T_p^{\mathbf{C}}M$$

which is a complex vector bundle over $M$ (cf. (3.4) of Chapter 5). At each point $p \in M$ the $2m$ vector fields on $U_\alpha$,

$$\left\{\frac{\partial}{\partial x_1^\alpha}, \frac{\partial}{\partial y_1^\alpha}, \dots, \frac{\partial}{\partial x_m^\alpha}, \frac{\partial}{\partial y_m^\alpha}\right\}$$

generate $T_pM$ over $\mathbf{R}$ and $T_p^{\mathbf{C}}M$ over $\mathbf{C}$.

A linear mapping $J : T_pM \to T_pM$ over $\mathbf{R}$ and its complex extension $J : T_p^{\mathbf{C}}M \to T_p^{\mathbf{C}}M$ over $\mathbf{C}$ are defined by

$$J\left(\frac{\partial}{\partial x_j^\alpha}\right) = \left(\frac{\partial}{\partial y_j^\alpha}\right), \qquad J\left(\frac{\partial}{\partial y_j^\alpha}\right) = -\left(\frac{\partial}{\partial x_j^\alpha}\right), \qquad j = 1, \dots, m,$$

which satisfy

$$J^2 = J \circ J = -\mathrm{id},$$

where id is the identity mapping. $J$ is called the **almost complex structure** and is a $(1,1)$ tensor field.

A $C^\infty$-mapping $\phi : M \to N$ between two complex manifolds $M$, $N$ is **holomorphic** if, denoting the almost complex structures by $J$, the differentiation $\phi_* : T_pM \to T_{\phi(p)}N$, for $p \in M$ satisfies

$$J \circ \phi_* = \phi_* \circ J.$$

Then if we take local coordinates $(z_1, \dots, z_m)$, $(w_1, \dots, w_n)$ of $p \in M$, $\phi(p) \in N$, and put

$$z_j = x_j + \sqrt{-1}\, y_j, \qquad w_k = u_k + \sqrt{-1}\, v_k, \qquad 1 \le j \le m,\ 1 \le k \le n,$$

then we have that

$\phi$ is holomorphic in a neighborhood of $p$.

$\iff$ Each $w_k \circ \phi$ is a holomorphic function in $(z_1, \dots, z_m)$, $1 \le k \le n$.

$\iff$ Each $u_k \circ \phi$ and $v_k \circ \phi$ satisfy the **Cauchy-Riemann** equations :

$$\frac{\partial(u_k \circ \phi)}{\partial x_j} = \frac{\partial(v_k \circ \phi)}{\partial y_j}, \qquad \frac{\partial(u_k \circ \phi)}{\partial y_j} = -\frac{\partial(v_k \circ \phi)}{\partial x_j},$$

for $1 \le j \le m$.

A Riemannian metric $g$ on a complex manifold $M$ is a **Hermitian metric** if

$$g(JX, JY) = g(X, Y), \qquad X, Y \in \mathfrak{X}(M),$$

and if the 2-form $\omega$ given by

$$\omega(X, Y) := g(X, JY), \qquad X, Y \in \mathfrak{X}(M)$$

is a closed form, i.e., $d\omega = 0$, $g$ is called a **Kähler metric** and $(M, g)$ is called a **Kähler manifold**.

It is known (cf. [K.N]) that a sufficient and necessary condition for a hermitian metric $g$ on a complex manifold $M$ to be a Kähler metric is that

$$\nabla_X(JY) = J(\nabla_X Y), \qquad X, Y \in \mathfrak{X}(M). \tag{3.13}$$

**PROPOSITION (3.14).** *A holomorphic mapping $\phi : M \to N$ between two Kähler manifolds $(M, g)$, $(N, h)$ is harmonic.*

**PROOF.** We can take a local orthonormal frame $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ in such a way that $Je_i = f_i$, $Jf_i = -e_i$, $1 \le i \le m$. Then we get

$$\begin{aligned} \tilde{\nabla}_{f_i}\phi_* f_i &= \tilde{\nabla}_{f_i} J\phi_* e_i && \text{(by holomorphicity of } \phi) \\ &= J\tilde{\nabla}_{f_i}\phi_* e_i && \text{(since } (N, h) \text{ Kähler)} \\ &= J(\tilde{\nabla}_{e_i}\phi_* f_i + \phi_*[f_i, e_i]) && \text{(by Lemma (1.16))} \\ &= -\tilde{\nabla}_{e_i}\phi_* e_i + J\phi_*[f_i, e_i], \end{aligned}$$

since $\phi$ is holomorphic and $(N, h)$ is Kähler. By the same way, we get

$$\begin{aligned} \phi_*\nabla_{f_i} f_i &= \phi_* J\nabla_{f_i} e_i && ((M, g) \text{ is Kähler}) \\ &= \phi_* J(\nabla_{e_i} f_i + [f_i, e_i]) && \text{(by Theorem (3.5) in Chapter 2)} \\ &= -\phi_*\nabla_{e_i} e_i + J\phi_*[f_i, e_i], \end{aligned}$$

since $\phi$ is holomorphic and $(M, g)$ is Kähler. Thus, we obtain

$$\sum_{i=1}^{m}(\tilde{\nabla}_{f_i}\phi_* f_i - \phi_*\nabla_{f_i} f_i) = -\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i} e_i)$$

which yields that $\tau(\phi) = 0$ everywhere on $M$. □

**REMARK.** See the next chapter for the stability of holomorphic mappings and related topics.

Before ending Chapter 4, we show well-known examples of Kähler manifolds.

(3.15) **Example** 7 (The complex Euclidean space). Let $(z_1, \dots, z_n)$ be the standard coordinate of $\mathbf{C}^n$ and put $z_j = x_j + \sqrt{-1}y_j$, $1 \le j \le n$, and define

$$g_0 = \mathrm{Re}\left(\sum_{j=1}^{n} dz_j \otimes d\overline{z}_j\right) = \sum_{j=1}^{n}(dx_j \otimes dx_j + dy_j \otimes dy_j),$$

then $(\mathbf{C}^n, g_0)$ is a Kähler manifold. The corresponding almost complex structure $J$ is

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

and the corresponding 2-form $\omega$ is given by

$$\omega = \frac{\sqrt{-1}}{4}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = -\frac{1}{2}\sum_{j=1}^{n} dx_j \wedge dy_j,$$

and satisfies $d\omega = 0$.

(3.16) **Example 8** (Complex torus). Take a basis $\{v_j\}_{j=1}^n$ of $\mathbf{C}^n$, and put

$$\Lambda := \left\{ \sum_{j=1}^n (m_j + \sqrt{-1} n_j) v_j ;\ m_j,\ n_j \in \mathbf{Z},\ 1 \le j \le n \right\}.$$

Two elements $z = \sum_{j=1}^n a_j v_j$, $w = \sum_{j=1}^n b_j v_j$ of $\mathbf{C}^n$ with $a_j, b_j \in \mathbf{C}$ are equivalent if $z - w \in \Lambda$, the set of all equivalence classes $\pi(z)$, $z \in \mathbf{C}^n$ is denoted by $\mathbf{C}^n/\Lambda$ which becomes a compact $n$-dimensional complex manifold. We can give a complex local coordinate of $\mathbf{C}^n/\Lambda$ by $\mathbf{C}^n/\Lambda \ni \pi(z) = \pi(\sum_{j=1}^n a_j v_j) \mapsto (a_1, \dots, a_n)$. We can give a Riemannian metric $g_\Lambda$ on $\mathbf{C}^n/\Lambda$ by

$$g_\Lambda = \mathrm{Re}\left( \sum_{i,j=1}^n \langle v_i, v_j \rangle\, da_i \otimes d\overline{a}_j \right).$$

Here $< z, w >= \sum_{i=1}^n z_i \overline{w}_i$, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbf{C}^n$. Then we get $\pi^* g_\Lambda = g_0$, where $\pi : \mathbf{C}^n \to \mathbf{C}^n/\Lambda$ is the projection. The resulting complex Hermitian manifold $(\mathbf{C}^n/\Lambda, g_\Lambda)$ is a Kähler manifold with zero sectional curvature.

In general, if we take a basis of $\mathbf{R}^{2n}$, $\{\tau_1, \dots, \tau_{2n}\}$, and put

$$\Lambda = \left\{ \sum_{i=1}^{2n} m_i \tau_i ;\ m_i \in \mathbf{Z} \right\},$$

then $\mathbf{C}^n/\Lambda$ also becomes a compact complex manifold and the Riemannian metric $g_\Lambda$ induced from $g_0$ is a Kähler metric whose sectional curvature is zero. This is called a **complex torus**.

(3.17) **Example 9** (Complex projective space). Let

$$\mathbf{C}^{n+1} := \{ z = {}^t(z_1, \dots, z_{n+1});\ z_i \in \mathbf{C},\ 1 \le i \le n+1 \},$$

where ${}^t$ means the transpose. Two elements $z = {}^t(z_1, \dots, z_{n+1})$, $w = {}^t(w_1, \dots, w_{n+1})$ in $\mathbf{C} - (0)$ are equivalent if $z = \lambda w$, i.e., $z_i = \lambda w_i$, $1 \le i \le n+1$, for some $\lambda \in \mathbf{C} - (0)$. The totality of all equivalence classes $[z]$, $z \in \mathbf{C}^{n+1} - (0)$ is denoted by $P^n(\mathbf{C})$ which becomes an $n$-dimensional complex manifold. For $1 \le i \le n+1$,

$$U_i := \{ [{}^t(z_1, \dots, z_{n+1})] \in P^n(\mathbf{C});\ z_i \ne 0 \},$$

and define an into homeomorphism by

$$\alpha_i : U_i \ni [{}^t(z_1, \dots, z_{n+1})] \mapsto \left( \frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i} \right) \in \mathbf{C}^n,$$

then $\{(U_i, \alpha_i);\ i = 1, \dots, n+1\}$ gives a complex coordinate system of $P^n(\mathbf{C})$.

On the other hand, a compact Lie group $G = \mathrm{SU}(n+1)$ acts transitively on $P^n(\mathbf{C})$ by

$$\mathrm{SU}(n+1) \times P^n(\mathbf{C}) \ni (x, [z]) \mapsto x \cdot [z] := [x \cdot z] \in P^n(\mathbf{C}).$$

Here $x \cdot z$ is the multiplication of a matrix $x$ and a vector $z$. The isotropy subgroup $K$ of $G$ at the origin $o = [{}^t(1, 0, \dots, 0)]$ is

$$K = S(\mathrm{U}(1) \times \mathrm{U}(n)) = \left\{ \begin{pmatrix} \alpha & {}^t\mathbf{0} \\ \mathbf{0} & x \end{pmatrix} ; \ \alpha \in \mathrm{U}(1),\ x \in \mathrm{U}(n),\ \alpha \det(x) = 1 \right\},$$

where $\mathbf{0}$ in the right-hand side is ${}^t(0, \dots, 0)$. Thus, it can be written as

$$P^n(\mathbf{C}) = \mathrm{SU}(n+1)/S(\mathrm{U}(1) \times \mathrm{U}(n)) = G/K,$$

and $P^n(\mathbf{C})$ is a compact homogeneous space. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of $G$, $K$, respectively, and define an inner product on $\mathfrak{g}$ by

$$\langle X, Y \rangle = -\mathrm{tr}(XY), \qquad X, Y \in \mathfrak{g}.$$

Then the orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to this inner product is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^t\overline{z} \\ z & O \end{pmatrix} ; \ z = {}^t(z_1, \dots, z_n),\ z_i \in \mathbf{C},\ 1 \le i \le n \right\}.$$

Let $h$ be an $G$-invariant Riemannian metric on $G/K = P^n(\mathbf{C})$ corresponding to the inner product

$$\langle X, Y \rangle_0 = \frac{1}{2} \langle X, Y \rangle, \qquad X, Y \in \mathfrak{m}.$$

Then $(P^n(\mathbf{C}), h)$ is a Kähler manifold whose sectional curvature varies over between 1 and 4.

On the other hand, let

$$S^{2n+1} := \{ z \in \mathbf{C}^{n+1} ;\ \|z\| = 1 \},$$

on which $\mathrm{SU}(n+1)$ acts transitively by

$$\mathrm{SU}(n+1) \times S^{2n+1} \ni (x, z) \mapsto x \cdot z \in S^{n+1}.$$

The isotropy subgroup $H$ at the origin $o = {}^t(1, 0, \dots, 0)$ is

$$H = \left\{ \begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{0} & x \end{pmatrix} ; \ x \in \mathrm{SU}(n) \right\} \cong \mathrm{SU}(n).$$

Thus, it can be written by

$$S^{n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n) = G/H.$$

$H$ is a closed Lie subgroup of $K$ and the natural projection

$$\pi : G/H \ni xH \mapsto xK \in G/K$$

coincides with the mapping

$$\pi : S^{n+1} \ni z \mapsto [z] \in P^n(\mathbf{C})$$

which is called the **Hopf mapping (fibration)**. Let $\mathfrak{n}$ be the orthogonal complement of the Lie algebra $\mathfrak{h}$ of $H$ in $\mathfrak{g}$ with respect to the inner product $\langle\,,\,\rangle$. This $\mathfrak{n}$ is given by

$$\mathfrak{n} = \left\{ \begin{pmatrix} in\theta & -{}^t\bar{z} \\ z & -i\theta I \end{pmatrix} ;\ \theta \in \mathbf{R},\ z = {}^t(z_1, \dots, z_n),\ z_i \in \mathbf{C} \right\} \supset \mathfrak{m}.$$

Here $I$ is the unit matrix of degree $n$. Let $g$ be an $G$-invariant Riemannian metric on $S^{n+1}$ corresponding to the inner product $\langle X, Y\rangle_0 = \frac{1}{2}\langle X, Y\rangle$, $X, Y \in \mathfrak{n}$. Then $g$ coincides with the Riemannian metric $g_{S^{2n+1}}$ with sectional curvature one and the Hopf mapping $\pi : (S^{2n+1}, g) \to (P^n(\mathbf{C}), h)$ is a Riemannian submersion, and moreover, a harmonic mapping (see also Exercise 4.5).

## Exercises

4.1. Show that any isometry $\phi: (M, g) \to (N, h)$ is a harmonic mapping.

4.2. Let $\dim(M) = 2$. For a $C^\infty$-mapping $\phi: (M, g) \to (N, h)$, its energy $E(\phi)$ does not change even when $g$ is changed into $\lambda g$, where $\lambda \in C^\infty(M)$ is a positive function.

4.3. Let $\dim(M) = 2$. Let $(x, y)$ be a local coordinate on a neighborhood $U$ in $M$ such that

$$g = \lambda(dx \otimes dx + dy \otimes dy),$$

where $\lambda \in C^\infty(U)$ is a positive function. For a harmonic mapping $\phi : (M, g) \to (N, h)$, show that the function $\psi$ on $U$ defined by

$$\psi := \left|\frac{\partial\phi}{\partial x}\right|^2 - \left|\frac{\partial\phi}{\partial y}\right|^2 - 2\sqrt{-1}\left\langle\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right\rangle$$

is holomorphic in $z = x + \sqrt{-1}y$ on $U$. Here

$$\frac{\partial\phi}{\partial x} = \phi_*\left(\frac{\partial}{\partial x}\right), \qquad \frac{\partial\phi}{\partial y} = \phi_*\left(\frac{\partial}{\partial y}\right),$$
$$\left|\frac{\partial\phi}{\partial x}\right|^2 = h\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x}\right), \qquad \left|\frac{\partial\phi}{\partial y}\right|^2 = h\left(\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial y}\right),$$
$$\left\langle\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right\rangle = h\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right).$$

4.4. (i) Let $(M_1 \times M_2, g_1 \times g_2)$ be the Riemannian product of two Riemannian manifolds $(M_i, g_i)$, $i = 1, 2$. Assume that a $C^\infty$-mapping

$$\phi : M_1 \times M_2 \ni (x_1, x_2) \mapsto \phi(x_1, x_2) \in N$$

is harmonic in each variable, i.e., $(M_i, g_i) \ni x_i \mapsto \phi(x_1, x_2) \in (N, h)$ are harmonic, $i = 1, 2$. Show that $\phi : (M_1 \times M_2, g_1 \times g_2) \to \phi(x_1, x_2) \in (N, h)$ is harmonic.

(ii) Let $G$ be a compact Lie group, and let $g$ be a bi-invariant Riemannian metric on $G$. Then, using (i), show that the multiplication defined by

$$\phi : G \times G \ni (x, y) \mapsto xy \in G$$

is harmonic.

(iii) Let $F : \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ be a bilinear mapping satisfying

$$\|F(x, y)\| = \|x\| \, \|y\|, \qquad x \in \mathbf{R}^p, \ y \in \mathbf{R}^q.$$

Then show by (i) that the mapping $\overline{F} : S^{p-1} \times S^{q-1} \to S^{n-1}$ naturally induced from $F$ is harmonic.

4.5. Let $( \, , \, )$ be the standard Hermitian inner product on $\mathbf{C}^2$ and $|x| = (x, x)^{1/2}$, $x \in \mathbf{C}^2$. Two elements $x, y \in \mathbf{C}^2 - (0)$ are equivalent if $x = \lambda y$, $\lambda \in \mathbf{C} - (0)$. The set of all equivalence classes $[x]$ is denoted by $P^1(\mathbf{C})$ which is diffeomorphic to $S^2$. Let $S^3 := \{x \in \mathbf{C}^2 ; \, |x| = 1\}$. The mapping

$$\phi : S^3 \ni x \mapsto [x] \in S^2 = P^1(\mathbf{C})$$

is called the Hopf mapping. Show that the Hopf mapping $\phi : (S^3, g_{S^3}) \to (S^2, g_{S^2})$ is harmonic.

≪ **Coffee Break** ≫ **Soap films and minimal surfaces** (Plateau's problem)

In 1760, J.L. Lagrange derived the famous equation of minimal surfaces and published this in 1762 in *Essay of a new method for determining the maxima and minima of an indefinite integral formula.*

111 years later, in 1873, J. Plateau, a physicist in Belgium, published *Experimental and theoretic statics of liquid influenced only by molecular forces.* This reported his interesting experiments about soup films. By his experiments, it was supported that "Any contour by a single closed wire, if it is not so large, is covered with at least one soup film."

The corresponding mathematical problem is "For any Jordan curve, i.e., continuous closed curve without self intersection in the 3-dimensional space, does there exists at least one minimal surface bounding this curve?" This probelm has been known as Plateau's problem. B. Riemann, K. Weierstrass and H.A. Schwarz gave special minimal surfaces for special type Jordan curves, but did not give a general solution.

In 1930, 1931, J. Douglas and T. Rado solved independently Plateau's problem. See J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), 263–321;

T. Rado, *On Plateau's problem*, Ann. of Math. **31** (1930), 457–469; and

T. Rado, *The problem of least area and the problem of Plateau*, Math. Zeit. **32** (1930), 763–796.

Here we explain the mathematical background briefly.

Let

$$B := \{w = (u, v) \in \mathbf{R}^2 ; \, |u|^2 + |v|^2 < 1\},$$

the unit disc, and let

$\Gamma$ : a Jordan curve in $\mathbf{R}^3$.

Then we write a $C^\infty$-mapping $X : B \to \mathbf{R}^3$ by

$$B \ni w \mapsto X(w) = (X_1(w), X_2(w), X_3(w)) \in \mathbf{R}^3,$$

and

$$X_u := X_*\left(\frac{\partial}{\partial u}\right), \qquad X_v := X_*\left(\frac{\partial}{\partial v}\right),$$

then the area $A(X)$ of a surface $X$ is given by

$$A(X) = \int_B \sqrt{|X_u|^2 |X_v|^2 - (X_u, X_v)^2}\, du\, dv,$$

where $(\ ,\ )$, $|\ |$ are the standard inner product, and the norm of $\mathbf{R}^3$. See Figure 4.5.

The condition for $X$ to bound a Jordan curve $\Gamma$ is that

(0) a continuous mapping $X|_{\partial B} : \partial B \to \Gamma$ gives a parameter of the Jordan curve $\Gamma$.

Then Plateau's problem is to find $X$ which minimizes the area $A$ satisfying the condition (0).

To do this, it suffices to find $X$ satisfying $\frac{d}{dt}|_{t=0} A(X_t) = 0$ for any variation $X_t$ satisfying the condition (0), and the equation of the first variation, which is called the Euler-Lagrange equation and says that the mean curvature of $X$ is zero. It is known that this is equivalent to

(i) $\Delta X = 0$, that is, $\Delta X_i = 0$, $i = 1, 2, 3$, where $\Delta = -(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})$, and

(ii) $|X_u|^2 - |X_v|^2 = 0$ and $(X_u, X_v) = 0$.

Weierstrass pointed out, using the condition (i) $\Delta X = 0$, putting

$$\Phi(w) := |X_u|^2 - |X_v|^2 - 2\sqrt{-1}(X_u, X_v),$$

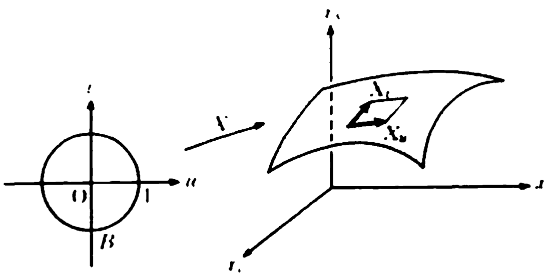


FIGURE 4.5. The area of the parallelogram spanned by $X_u$, $X_v$ is $\sqrt{|X_u|^2|X_v|^2 - (X_u, X_v)^2}$.

$\Phi(w)$ is a holomorphic function in $w = u + \sqrt{-1}\,v$ (see also Exercise 4.3), it has been a useful fact, but could not solve (0), (i), (ii).

The reason for this difficulty is that, for any diffeomorphism $\varphi : \overline{B} \to \overline{B}$ satisfying $\varphi(\partial B) = \partial B$,

(iii) $A(X \circ \varphi) = A(X)$.

So it implies that there are too many $X$ which minimize $A$. Douglas and Rado considered, instead of $A$, the energy (the Dirichlet integral)

$$E(X) = \frac{1}{2}\int_B |dX|^2\,du\,dv = \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2)\,du\,dv.$$

This $E$ has the following good properties:

(iv) $A(X) \le E(X)$, and the equality holds if and only if $|X_u|^2 - |X_v|^2 = 0$ and $\langle X_u, X_v\rangle = 0$.

(v) The only diffeomorphism $\varphi : \overline{B} \to \overline{B}$ with $\varphi(\partial B) = \partial B$ which satisfies

$$E(X \circ \varphi) = E(X)$$

is the one satisfies that

$$|\varphi_u|^2 - |\varphi_v|^2 = 0 \text{ and } \langle \varphi_u, \varphi_v\rangle = 0 \quad (\text{on } \overline{B}).$$

(vi) If $X$ minimizes $E$, then it satisfies (i), (ii) and is smooth on $B$.

It is not too hard to check (iv). In fact, it follows from

$$\begin{aligned} A(X) &= \int_B \sqrt{|X_u|^2|X_v|^2 - \langle X_u, X_v\rangle^2}\,du\,dv \\ &\le \int_B |X_u|\,|X_v|\,du\,dv \\ &\le \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2)\,du\,dv = E(X), \end{aligned}$$

from which comes the equality case.

Thus, they suceeded in finding an $X_0$ which minimizes $E$ and simultaneously $A$ for a given Jordan curve $\Gamma$.

For topics and results on Plateau's problem, see

S. Hildebrandt and A. Tromba, *Mathematics and optimal forms*, Scientific American Books Inc., 1985.

M. Struwe, *Plateau's Problem and the calculus of variations*, Princeton Univ. Press, Vol. 35, Princeton, NJ, 1988.

F. Morgan, *Geometric measure theory, A beginner's guide*, Academic Press, San Diego, CA, 1988.

CHAPTER 5

# The Second Variation Formula and Stability

In Chapter 5, we calculate the second variation formula of the energy (the action integral), and discuss the stability or the unstability of a harmonic mapping, and its rigidity.

The second variation formula was obtained independently by E. Mazet and R.T. Smith, in 1973–1975. The stability of holomorphic mappings between Kähler manifolds was obtained by A. Lichnerowicz in 1970. In this chapter, we discuss its relation to the second variation formula, which is useful to study the structure of the set of holomorphic mappings between Kähler manifolds.

In contrast to the above, there is an instability theorem which was found by Y.L. Xin in 1980. Recently, it was generalized by Y. Ohnita and R. Howard-S.W. Wei independently.

## §1. The second variation formula

**1.1. Calculation of the formula.** In Chapter 4, we derived in two different ways, the equation (the Euler-Lagrange equation) of critical points of the energy (the action integral) $E$ on $C^\infty(M, N)$. In this section, we calculate the second variation formula of $E$.

Let $\phi : (M, g) \to (N, h)$ be a harmonic mapping. Take a smooth variation $\phi_{s,t} : M \to N$ with two parameters $s$, $t$, and with $\phi_{0,0} = \phi$. That is, the mapping $F$ defined by

$$F : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M \ni (s, t) \mapsto \phi_{s,t}(x) \in N$$

is a $C^\infty$-mapping and $F(0, 0, x) = \phi(x)$, $x \in M$. The corresponding variation vector fields are denoted by $V$, $W$: For $x \in M$, define

$$V(x) := \frac{d}{dt}\Big|_{s=0} \phi_{s,0}(x) \in T_{\phi(x)}N, \qquad W(x) := \frac{d}{dt}\Big|_{t=0} \phi_{0,t}(x) \in T_{\phi(x)}N.$$

We often extend vector fields on the interval $(-\epsilon, \epsilon)$ or $M$ to the ones on the product manifold $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M$, and we denote them also by the same letter as $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial t}$, $X$, for $X \in \mathfrak{X}(M)$. Then by definition, we get

$$V = F_*\left(\frac{\partial}{\partial s}\right)_{(s,t)=(0,0)}, \qquad W = F_*\left(\frac{\partial}{\partial t}\right)_{(s,t)=(0,0)}.$$

Now we regard the energy $E$ to be a function on a manifold $C^\infty(M, N)$, the **Hessian** of $E$, $H(E)_\phi(V, W)$, at a critical point $\phi$, a harmonic mapping, is defined as follows (cf. §1 in Chapter 3):

$$H(E)_\phi(V, W) = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E(\phi_{s,t}). \tag{1.1}$$

In the following, we shall calculate (1.1). By definition,

$$\begin{aligned} E(\phi_{s,t}) &= \frac{1}{2}\int_M \sum_{i=1}^m h(\phi_{s,t*}e_i, \phi_{s,t*}e_i)\, v_g \\ &= \frac{1}{2}\int_M \sum_{i=1}^m h(F_* e_i, F_* e_i)\, v_g. \end{aligned}$$

Repeating the calculation in §1 of Chapter 4, we get the same equation as (1.18) in Chapter 4,

$$\frac{\partial}{\partial t} E(\phi_{s,t}) = -\int_M h\left(F_* \frac{\partial}{\partial t}, \sum_{i=1}^m \{\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\}\right) v_g. \tag{1.2}$$

By differentiating it in the variable $s$, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} E(\phi_{s,t}) &= -\int_M \frac{\partial}{\partial s} h\left(F_* \frac{\partial}{\partial t}, \sum_{i=1}^m \{\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\}\right) v_g \\ &= -\int_M h\left(\tilde{\nabla}_{\frac{\partial}{\partial s}} F_* \frac{\partial}{\partial t}, \sum_{i=1}^m \{\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\}\right) v_g \\ &\quad - \int_M h\left(F_* \frac{\partial}{\partial t}, \sum_{i=1}^m \tilde{\nabla}_{\frac{\partial}{\partial s}} \{\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\}\right) v_g. \end{aligned}$$

Here in the last equality, we used the compatibility condition of $\tilde{\nabla}$ anf $h$ (cf. (1.12) in Chapter 4).

The first term of the above vanishes at $(s, t) = (0, 0)$. Because in the integrand,

$$\sum_{i=1}^m \{\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\} = \tau(\phi) = 0,$$

since $\phi$ is harmonic. So we calculate the integrand of the second term of the above, and it follows that

$$\tilde{\nabla}_{\frac{\partial}{\partial s}} \tilde{\nabla}_{e_i} F_* e_i = \tilde{\nabla}_{e_i} \tilde{\nabla}_{\frac{\partial}{\partial s}} F_* e_i + \tilde{\nabla}_{[\frac{\partial}{\partial s}, e_i]} F_* e_i + {}^N R\left(F_* \frac{\partial}{\partial s}, F_* e_i\right) F_* e_i \tag{1.3}$$

which follows from the definition of the curvature tensor ${}^N R$. Note that the third term of (1.3) has meaning because ${}^N R$ is determined only by the values at $T_y N$ at each point $y \in N$ by (3.17) in Chapter 2.

Furthermore, apply Lemma(1.16) in Chapter 4 to $\tilde{\nabla}_{\frac{\partial}{\partial s}} F_* e_i$. Then since $[\frac{\partial}{\partial s}, e_i] = 0$, (1.3) coincides with

$$\tilde{\nabla}_{e_i}\left(\tilde{\nabla}_{e_i} F_* \frac{\partial}{\partial s} + F_*\left[\frac{\partial}{\partial s}, e_i\right]\right) + \tilde{\nabla}_{[\frac{\partial}{\partial s}, e_i]} F_* e_i + {}^N R\left(F_* \frac{\partial}{\partial s}, F_* e_i\right) F_* e_i$$
$$= \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} F_* \frac{\partial}{\partial s} + {}^N R\left(F_* \frac{\partial}{\partial s}, F_* e_i\right) F_* e_i.$$

Making use of Lemma (1.16) in Chapter 4, we get

$$\tilde{\nabla}_{\frac{\partial}{\partial s}} F_* \nabla_{e_i} e_i = \tilde{\nabla}_{\nabla_{e_i} e_i} F_* \frac{\partial}{\partial s} + F_*\left[\frac{\partial}{\partial s}, \nabla_{e_i} e_i\right] = \tilde{\nabla}_{\nabla_{e_i} e_i} F_* \frac{\partial}{\partial s}. \tag{1.4}$$

Finally, putting $(s, t) = (0, 0)$, since $F_* \frac{\partial}{\partial s} = V$, $F_* \frac{\partial}{\partial t} = W$, and $F_* e_i = \phi_* e_i$, we obtain the following theorem:

**THEOREM** (1.5) (the second variation formula). *Let $\phi: (M, g) \to (N, h)$ be a harmonic mapping. Then the Hessian of the energy $E$ at $\phi$ is given by*

$$H(E)_\phi(V, W) = \int_M h(J_\phi(V), W) v_g, \qquad V, W \in \Gamma(\phi^{-1}TN). \tag{1.6}$$

*Here $J_\phi$ is a second order selfadjoint elliptic differential operator acting on the space of variation vector fields along $\phi$, $\Gamma(\phi^{-1}TN)$ of the form:*

$$J_\phi(V) := -\sum_{i=1}^m (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i}) V - \sum_{i=1}^m {}^N R(V, \phi_* e_i) \phi_* e_i, \tag{1.7}$$

*for $V \in \Gamma(\phi^{-1}TN)$.*

**DEFINITION** (1.8). The operator $J_\phi := \overline{\Delta}_\phi - \mathscr{R}_\phi$ is called the **Jacobi operator**. Here the operator $\overline{\Delta}_\phi$ is defined by

$$\overline{\Delta}_\phi V := -\sum_{i=1}^m (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i}) V, \qquad V \in \Gamma(\phi^{-1}TN) \tag{1.9}$$

called the **rough Laplacian**. $\overline{\Delta}_\phi$ is a second-order elliptic differential operator because of the form (1.9). $\mathscr{R}_\phi$ is given by

$$\mathscr{R}_\phi V := \sum_{i=1}^m {}^N R(V, \phi_* e_i) \phi_* e_i, \qquad V \in \Gamma(\phi^{-1}TN). \tag{1.10}$$

Since $\mathscr{R}_\phi(fV) = f \mathscr{R}_\phi(V)$, $f \in C^\infty(M)$, $\mathscr{R}_\phi$ induces a bundle mapping of $E = \phi^{-1}TN$ defined by

$$\mathscr{R}_{\phi, x} : T_{\phi(x)}N \ni v \mapsto \sum_{i=1}^m {}^N R(v, \phi_* e_i) \phi_* e_i \in T_{\phi(x)}N,$$

which satisfies $(\mathscr{R}_\phi V)(x) = \mathscr{R}_{\phi, x}(V(x))$ for $x \in M$. Moreover,

$$h(\mathscr{R}_\phi V, W) = h(V, \mathscr{R}_\phi W), \qquad V, W \in \Gamma(\phi^{-1}TN). \tag{1.11}$$

**PROPOSITION (1.12).** *The rough Laplacian $\overline{\Delta}_\phi$ satisfies*

$$\int_M h(\overline{\Delta}_\phi V, W)\, v_g = \int_M h(\tilde{\nabla} V, \tilde{\nabla} W)\, v_g = \int_M h(V, \overline{\Delta}_\phi W)\, v_g, \qquad (1.13)$$

*for $V, W \in \Gamma(\phi^{-1}TN)$. Here $\tilde{\nabla} V$ is a $C^\infty$-section of the vector bundle $\phi^{-1}TN \otimes T^*M$ and is defined by*

$$\tilde{\nabla} V : \mathfrak{X}(M) \ni X \mapsto \tilde{\nabla}_X V \in \Gamma(\phi^{-1}TN),$$

*and for $W \in \Gamma(\phi^{-1}TN)$,*

$$h(\tilde{\nabla} V, \tilde{\nabla} W) = \sum_{i=1}^m h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{e_i} W) \in C^\infty(M).$$

**PROOF.** It is enough to show the first equation of (1.13). Since $\tilde{\nabla}$ and $h$ are compatible,

$$\begin{aligned} h(\overline{\Delta}_\phi V, W) = &- \sum_{i=1}^m \{e_i \cdot h(\tilde{\nabla}_{e_i} V, W) - h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{e_i} W)\} \\ &+ \sum_{i=1}^m h(\tilde{\nabla}_{\nabla_{e_i} e_i} V, W). \end{aligned}$$

Here define $X \in \mathfrak{X}(M)$ by

$$g(X, Y) = h(\tilde{\nabla}_Y V, W), \qquad Y \in \mathfrak{X}(M),$$

then we get

$$\begin{aligned} \operatorname{div}(X) &= \sum_{i=1}^m g(e_i, \nabla_{e_i} X) = \sum_{i=1}^m \{e_i \cdot g(e_i, X) - g(\nabla_{e_i} e_i, X)\} \\ &= \sum_{i=1}^m \{e_i \cdot h(\tilde{\nabla}_{e_i} V, W) - h(\tilde{\nabla}_{\nabla_{e_i} e_i} V, W)\}. \end{aligned}$$

By Green's formula $\int_M \operatorname{div}(X)\, v_g = 0$, and we obtain the desired equation. □

**1.2. The index, nullity, and weak stability.** Due to Theorem (1.5) (the second variation formula), we can define the notions of the index, nullity, weak stability, and unstability for harmonic mappings in a similar way as for the Morse theory in Chapter 3.

**DEFINITION (1.14).** The **index** of a harmonic mapping $\phi : (M, g) \to (N, h)$ is

$$\sup\{\dim(F);\ F \subset \Gamma(\phi^{-1}TN),\ \text{a subspace on which } H(E)_\phi \text{ is negative definite}\},$$

denoted by index $(\phi)$. The **nullity** is

$$\dim\{V \in \Gamma(\phi^{-1}TN);\ H(E)_\phi(V, W) = 0, \text{ for all } W \in \Gamma(\phi^{-1}TN)\},$$

denoted by nullity$(\phi)$. A harmonic mapping $\phi$ is said to be **weakly stable** if

$$\mathrm{index}(\phi) = 0, \quad \text{i.e., } H(E)_\phi(V, V) \geq 0, \text{ for all } V \in \Gamma(\phi^{-1}TN),$$

and otherwise, is said to be **unstable**.

In the second variation formula, since $J_\phi$ is a selfadjoint ellptic differential operator acting on the space of sections of the vector bundle $\phi^{-1}TN$ over a compact manifold $M$, due to Hodge-de Rham-Kodaira theory, the spectrum of $J_\phi$, denoted by Spect$(J_\phi)$, consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities and without accumulation points. So we count the eigenvalues with their multiplicities, denoted as

$$\lambda_1(\phi) \leq \lambda_2(\phi) \leq \cdots \leq \lambda_i(\phi) \leq \cdots \uparrow \infty.$$

Here $\lambda$ is the eigenvalue of $J_\phi$ if

$$V_\lambda(\phi) := \{V \in \Gamma(\phi^{-1}TN);\ J_\phi V = \lambda V\} \neq \{0\},$$

which is called the **eigenspace** with the eigenvalue $\lambda$, $\dim V_\lambda(\phi)$ is called the **multiplicity** of the eigenvalue $\lambda$. As above, the index and nullity of a harmonic mapping $\phi: (M, g) \to (N, h)$ are given by

$$\mathrm{index}(\phi) = \sum_{\lambda<0} \dim V_\lambda(\phi), \tag{1.15}$$

$$\mathrm{nullity}(\phi) = \dim V_0(\phi) = \dim \mathrm{Ker}(J_\phi), \tag{1.16}$$

and

$$\phi \text{ is weakly stable } \Longleftrightarrow \lambda_i(\phi) \geq 0, \text{ for } i = 1, 2, \ldots. \tag{1.17}$$

If $(N, h)$ has the nonpositive sectional curvature, that is, for any linearly independent tangent vectors $u$, $v$ at any point $y \in N$,

$$^NK(u, v) \leq 0, \quad \text{i.e., } h(^NR(u, v)v, u) \leq 0,$$

then $h(\mathscr{R}_\phi V, V) \leq 0$, for all $V \in \Gamma(\phi^{-1}TN)$. Also we get

$$\int_M h(\overline{\Delta}_\phi V, V)\, v_g = \int_M h(\tilde{\nabla} V, \tilde{\nabla} V)\, v_g \geq 0, \tag{1.13}$$

due to (1.7), we obtain

$$\int_M h(J_\phi V, V)\, v_g \geq 0, \quad \text{for all } V \in \Gamma(\phi^{-1}TN).$$

Thus, we obtain

PROPOSITION 1.18. *If $(N, h)$ has nonpositive sectional curvature, then any harmonic mapping $\phi: (M, g) \to (N, h)$ is weakly stable.*

REMARK (geometric meaning of the nullity). Let $(M, g)$, $(N, h)$ be arbitrary compact Riemannian manifolds, and let the totality of all harmonic mappings between them be denoted by

$$\mathfrak{Har}(M, N) := \{\phi: (M, g) \to (N, h), \text{ harmonic mappings}\}.$$

Then it is, in general, not a "manifold", but let us consider its tangent space at $\phi \in \mathfrak{Har}(M, N)$,

$$T_\phi \mathfrak{Har}(M, N) \subset T_\phi C^\infty(M, N).$$

It is natural to say that a variation vector field $V \in T_\phi C^\infty(M, N) = \Gamma(\phi^{-1}TN)$ belongs to $T_\phi \mathfrak{Har}(M, N)$ if there exists a one parameter family of harmonic mappings $\phi_s \in \mathfrak{Har}(M, N)$, $-\epsilon < s < \epsilon$ with $\phi_0 = \phi$ such that

$$V(x) = \frac{d}{ds}\Big|_{s=0} \phi_s(x), \qquad x \in M.$$

In this case, we obtain

$$T_\phi \mathfrak{Har}(M, N) \subset \operatorname{Ker}(J_\phi), \tag{1.18'}$$

in particular,

$$\dim T_\phi \mathfrak{Har}(M, N) \leq \operatorname{nullity}(\phi). \tag{1.18''}$$

PROOF. In fact, let $\phi_s \in \mathfrak{Har}(M, N)$ be a one parameter family of harmonic mappings with $\phi_0 = \phi$, and let $\phi_{s,t} \in C^\infty(M, N)$ be any variation of $\phi_s$, $-\epsilon < t < \epsilon$, with $\phi_{s,0} = \phi_s$. Define

$$V(x) := \frac{d}{ds}\Big|_{s=0} \phi_{s,0}(x), \qquad W(x) := \frac{d}{dt}\Big|_{t=0} \phi_{0,t}(x), \qquad x \in M.$$

Then since the $\phi_s$ are harmonic for all $s$, we obtain

$$\frac{\partial}{\partial t}\Big|_{t=0} E(\phi_{s,t}) = 0.$$

Therefore, we obtain

$$\int_M h(J_\phi(V), W)\, v_g = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E(\phi_{s,t}) = 0.$$

Since $W \in \Gamma(\phi^{-1}TN)$ is arbitrary, we get $V \in \operatorname{Ker}(J_\phi)$, and we obtain (1.18'). □

In general, the equality $T_\phi \mathfrak{Har}(M, N) = \operatorname{Ker}(J_\phi)$ is false. However, by the quantity nullity $(\phi) = \dim \operatorname{Ker}(J_\phi)$, we know how a finite dimensional neighborhood in $\mathfrak{Har}(M, N)$ spreads in an infinite dimensional neighborhood in $C^\infty(M, N)$ for a harmonic mapping $\phi$. It is very important to determine the structure of a "finite dimensional submanifold" $\mathfrak{Har}(M, N)$ of an infinite dimensional manifold $C^\infty(M, N)$.

**1.3. Examples.** We discuss the second variation formula for the simplest cases.

(1.19) EXAMPLE 1 (constant mappings). For any compact Riemannian manifolds, $(M, g)$, $(N, h)$, consider the second variation of a constant mapping

$$\phi : M \to N, \qquad \phi(x) = q, \quad \text{for all } x \in M.$$

By definition,

$$\Gamma(\phi^{-1}TN) = \{V;\ V(x) \in T_qN,\ \forall x \in M\}.$$

Thus, letting $\{v_i\}_{i=1}^n$ be a basis of $T_qN$, we may define $V_i \in \Gamma(\phi^{-1}TN)$, $1 \le i \le n$, by

$$V_i(x) := v_i, \qquad x \in M.$$

Since any element in $T_qN$ can be expressed by a linear combination of $\{v_i\}_{i=1}^n$, we get

$$\Gamma(\phi^{-1}TN) = \left\{\sum_{i=1}^n f_i V_i;\ f_i \in C^\infty(M),\ 1 \le i \le n\right\}.$$

Then we can give a variation $\phi_t$ of a constant mapping $\phi$ corresponding to a variation vector field $V = \sum_{i=1}^n f_i V_i$, by

$$\phi_t(x) = \exp_q\left(t \sum_{i=1}^n f_i(x)\, v_i\right), \qquad x \in M.$$

Let us calculate the Jacobi operator $J_\phi$.

(i) For $V \in \Gamma(\phi^{-1}TN)$, since $\phi_* e_i = 0$, $1 \le i \le n$, we get

$$\mathscr{R}_\phi(V) = \sum_{i=1}^m {}^N R(V, \phi_* e_i)\phi_* e_i = 0.$$

(ii) On the other hand, for $X \in \mathfrak{X}(M)$ and $V = \sum_{i=1}^n f_i V_i \in \Gamma(\phi^{-1}TN)$, by (1.11) in Chapter 4 and the properties of the covariant differentiation,

$$\begin{aligned}\tilde{\nabla}_X V &= \sum_{i=1}^n \{Xf_i \cdot V_i + f_i \tilde{\nabla}_X V_i\} \\ &= \sum_{i=1}^n Xf_i \cdot V_i,\end{aligned}$$

since $\tilde{\nabla}_X V_i = 0$ by means of $V_i(x) = v_i$, for all $x \in M$.

Therefore, by (i), (ii), we obtain

$$J_\phi V = \sum_{i=1}^n (\Delta_g f_i)\, V_i, \qquad V = \sum_{i=1}^n f_i V_i \in \Gamma(\phi^{-1}TN).$$

Here $\Delta_g$ is the Laplacian acting on $C^\infty(M)$ (cf. (3.28) in Chapter 2) which is of the form

$$\Delta_g f = -\sum_{i=1}^m \left\{e_i(e_i f) - \nabla_{e_i} e_i f\right\}, \qquad f \in C^\infty(M).$$

Thus, we obtain

PROPOSITION (1.20). *For a constant mapping* $\phi : (M, g) \to (N, h)$, *the spectrum* $\mathrm{Spec}(J_\phi)$ *of the Jacobi operator* $J_\phi$ *is the set of eigenvalues of the Laplacian of* $(M, g)$ *acting on* $C^\infty(M)$ *counted* ($n = \dim N$)*-times*:

$$\mathrm{Spec}(J_\phi) = n \times \mathrm{Spec}(\Delta_g), \qquad n = \dim(N).$$

*In particular,* $\mathrm{Spec}(J_\phi)$ *does not depend on* $q = \phi(M) \in N$.

REMARK. As above, the second variation is nontrivial even for a constant mapping. By Proposition (1.20), the studies of the Jacobi operator $J_\phi$ are the natural extensions of the ones of the spectrum of the Laplacian $\Delta_g$ of a Riemannian manifold $(M, g)$ (cf. [Ur 10]).

We denote the spectrum $\mathrm{Spec}(\Delta_g)$ of $\Delta_g$ by

$$\mathrm{Spec}(\Delta_g) = \{\lambda_0(g) = 0 < \lambda_1(g) \le \lambda_2(g) \le \cdots \le \uparrow \infty\}.$$

Here the eigenvalue $\lambda_0(g) = 0$ corresponds to the constant function on $M$. By Proposition (1.20), the index and nullity of a constant mapping $\phi$ are

$$\mathrm{index}(\phi) = 0, \quad \mathrm{nullity}(\phi) = \dim N.$$

(1.21) EXAMPLE 2 (the identity mapping). The identity mapping of a compact Riemannian manifold $(M, g)$,

$$\phi = \mathrm{id} : M \ni x \mapsto x \in M$$

is a trivial example of a harmonic mapping. But in this case, the theory of the second varition is much more complicated. Indeed, in this case,

$$M = N, \quad \Gamma(\phi^{-1}TN) = \mathfrak{X}(M),$$

and the Jacobi operator $J_{\mathrm{id}} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is given by

$$J_{\mathrm{id}} = \bar{\Delta} - \rho. \tag{1.22}$$

Here the operators $\bar{\Delta}$ and $\rho$ are

$$\bar{\Delta}(X) := -\sum_{i=1}^{m}(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})X, \tag{1.23}$$

$$\rho(X) := \sum_{i=1}^{m} R(X, e_i)e_i, \tag{1.24}$$

for $X \in \mathfrak{X}(M)$.

The Jacobi operator $J_{\mathrm{id}}$ is related to the Laplacian $\Delta_1$ acting on the space $A^1(M)$ of 1-forms on $M$ (cf. (3.36) in Chapter 2) as follows: 1-forms $\omega \in A^1(M)$ and vector fields $X \in \mathfrak{X}(M)$ on $M$ correspond to each other isomorphically in such a way that

$$\omega(Y) = g(X, Y), \qquad Y \in \mathfrak{X}(M). \tag{1.25}$$

Under this identification, we obtain the operator $\Delta_H : \mathfrak{X}(M) \to \mathfrak{X}(M)$ acting on vector fields corresponding to the Laplacian $\Delta_1 : A^1(M) \to A^1(M)$ acting on 1-forms. Then we obtain

THE WEITZENBÖCK FORMULA (1.26). *Under the above situation,*

$$\Delta_H = \overline{\Delta} + \rho.$$

PROOF. The proof will be given in the remark following (2.9) in a more general setting. □

By the formula in (1.26), we obtain

$$J_{\mathrm{id}} = \Delta_H - 2\rho. \tag{1.27}$$

REMARK. Vector fields belonging to $\mathrm{Ker}(J_{\mathrm{id}})$ were studied initially by T. Nagano and K. Yano in 1961, [N.Y 1], [N.Y 2]. They called them **geodesic vector fields.**

We write here without proof, the following theorem which is obtained from (1.27). See Exercise 5.1.

THEOREM (1.28) (R.T. Smith [St 1]). *Let $(M, g)$ be a compact Riemannian manifold which is* **Einstein**, *i.e., its Ricci tensor $\rho$ (cf. (3.19) in Chapter 2) satisfies*

$$\rho(X, Y) = c\, g(X, Y), \qquad X, Y \in \mathfrak{X}(M).$$

*Then:*

(i) *The identity mapping of $M$*, id: $(M, g) \to (M, g)$ *is weakly stable if and only if the first eigenvalue of the Laplacian $\Delta_g$ acting on $C^\infty(M)$, $\lambda_1(g)$, satisfies*

$$\lambda_1(g) \geq 2c.$$

(ii) *The nullity of the identity of $(M, g)$ is given as*

$$\text{nullity(id)} = \dim \mathrm{Iso}(M, g) + \dim\{f \in C^\infty(M);\ \Delta_g f = 2cf\},$$

*where* $\mathrm{Iso}(M, g)$ *is the isometry group of* $(M, g)$, i.e.,

$$\mathrm{Iso}(M, g) := \{\varphi;\ \varphi^* g = g\}.$$

A simply connected Riemannian manifold $(M, g)$ is called a **Riemannian symmetric space** if its curvature tensor $R$ satisfies $\nabla R = 0$. Then we obtain:

COROLLARY (1.29)(cf. [Ur 4], [Oh]). *Let $(M, g)$ be a compact simply connected irreducible Riemannian symmetric space. Then an $(M, g)$ for which the identity mapping is unstable is one of the following:*

(1) $S^n$, $n \geq 3$, *the unit sphere of dimension higher than or equal to* 3,

(2) $G_{\ell,q}(\mathbf{H}) = \mathrm{Sp}(\ell)/Sp(\ell - q) \times \mathrm{Sp}(q)$, $\ell - q \geq q \geq 1$, *the quaternionic Grassmann manifold,*

(3) $P^2(Cay) = F_4/\mathrm{Spin}(9)$, *the Cayley projective plane,*

(4) $E_6/F_4$,

(5) $\mathrm{SU}(2q)/\mathrm{Sp}(q)$, $q \geq 2$,

(6) $\mathrm{SU}(\ell + 1)$, $\ell \geq 1$, *a simple Lie group of type $A_\ell$, and*

(7) $\mathrm{Sp}(\ell)$, $\ell \geq 2$, *a simple Lie group of type* $C_\ell$.

(1.30) EXAMPLE 3 (closed geodesics). For the case of $M = S^1$ and $\phi$ : $(M, g) \to (N, h)$ a closed geodesic in $(N, h)$ of period one, i.e.,

$$\phi : [0, 1] \to (N, h), \qquad \phi(x+1) = \phi(x), \qquad -\infty < x < \infty,$$

then $\Gamma(\phi^{-1}TN)$ becomes the space of all $C^\infty$ vector fields $V$ along $\phi$, satsifying that $V(x) \in T_{\phi(x)}N$ and $V(x+1) = V(x)$ for all $x \in \mathbf{R}$. Then the Jacobi operator $J_\phi$ is

$$J_\phi V = -{}^N\nabla_{\phi'}{}^N\nabla_{\phi'}V - {}^NR(V, \phi')\phi'.$$

The vector fields satisfying $J_\phi = 0$ are called the **Jacobi fields** and play important rolls in the study of geodesics.

## §2. Instability theorems

**2.1. Main Theorems.** In this section, we prove the following theorem

THEOREM (2.1) (instability theorem, Y. L. Xin, 1980). *Let* $(S^m, g_{S^m})$ *be the unit sphere with constant sectional curvature one, with* $m \geq 3$. *Let* $(N, h)$ *be any compact Riemannian manifold. Then any non-constant harmonic mapping* $\phi : (S^m, g_{S^m}) \to (N, h)$ *is always unstable,* i.e., index$(\phi) > 0$.

This theorem was extended independently by Y. Ohnita, Y. L. Xin, and R. Howard-S.W. Wei into the following:

THEOREM (2.2). *Let* $(M, g)$ *be one of the Riemannian manifolds* (1)–(7) *in the list of Corollary* (1.29), *and let* $(N, h)$, $(M', g')$ *be arbitrary compact Riemannian manifolds. Then both nonconstant harmonic mappings*

$$\phi : (M, g) \to (N, h), \qquad \psi : (M', g') \to (M, g)$$

*are always unstable,* i.e., index$(\phi) > 0$ *and* index$(\psi) > 0$.

REMARK. B. White [Wh] showed (see also [Mn], [E.S, p. 130] ) the following theorem: Assume that a compact Riemannian manifold $(M, g)$ satisfies

$$\pi_1(M) = \pi_2(M) = 0.$$

Then for any two compact Riemannian manifolds $(M', g')$, $(N, h)$ and any continuous mappings $\psi : M' \to M$, $\phi : M \to N$,

$$\inf\{E(\eta);\ \eta \in C^\infty(M', M),\ \text{being homotopic to } \psi\} = 0,$$

and

$$\inf\{E(\eta);\ \eta \in C^\infty(M, N),\ \text{being homotopic to } \phi\} = 0.$$

The Riemannian manifolds listed in Corollary (1.29) and compact simply connected Lie groups have the property that both the first and second homotopy groups vanish. So it seems to be natural to consider the problem for compact Riemannian manifolds $(M, g)$ with $\pi_1(M) = \pi_2(M) = 0$, the similar assertions as Theorem (2.2) hold.

**2.2. Vector bundle valued differential forms.** Before going on to prove Theorem (2.1), we prepare the fundamental materials on vector bundle valued differential forms and the Weitzenböck formula for vector bundle valued 1-forms.

Let $E$ be a vector bundle over an $m$-dimensional Riemannian manifold $(M, g)$ on which $h$ is an inner product, $\tilde{\nabla}$ is a connection compatible with $h$. Let $\bigwedge^r T^*M \otimes E$ be the tensor bundle of $E$ and the $\bigwedge^r T^*M$. Let $A^r(E) = \Gamma(\bigwedge^r T^*M \otimes E)$ be the space of all $C^\infty$-sections of it, the elements of which are called **E-valued r-forms** on $M$ because $\omega(u_1, \dots, u_r) \in E_x$ if $u_1, \dots, u_r \in T_xM$ and $\omega \in A^r(E)$.

In the same way as for $A^r(M)$, we can give the following definitions:

Using $\tilde{\nabla}$, define the **exterior differentiation** $d^{\tilde{\nabla}} : A^r(E) \ni \omega \mapsto d^{\tilde{\nabla}}\omega \in A^{r+1}(E)$ by

$$d^{\tilde{\nabla}}\omega(X_1, \dots, X_{r+1}) \tag{2.3}$$
$$:= \sum_{i=1}^{r+1}(-1)^{i+1}\tilde{\nabla}_{X_i}(\omega(X_1, \dots, \hat{X}_i, \dots, X_{r+1}))$$
$$+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{r+1})$$

for $X_1, \dots, X_{r+1} \in \mathfrak{X}(M)$. Here $\hat{X}_i$ means to delete $X_i$ in the equation. In general, $d^{\tilde{\nabla}}(d^{\tilde{\nabla}}\omega)$ does not vanish, different from the case $A^r(M)$.

On the vector bundle $\bigwedge^r T^*M \otimes E$, the inner product is induced from the one $h$ on $E$ and the Riemannian metric $g$. We denote this inner product on each fiber $\bigwedge^r T_x^*M \otimes E_x$, $x \in M$, by $\langle\ ,\ \rangle_x$, $x \in M$. And we give the global inner product $(\ ,\ )$ on $A^r(E)$ by

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle v_g, \qquad \omega, \eta \in A^r(E).$$

Then the **co-differentiation** $\delta^{\tilde{\nabla}} : A^{r+1}(E) \to A^r(E)$ of $d^{\tilde{\nabla}}$ is the operator satisfying

$$(d^{\tilde{\nabla}}\omega, \eta) = (\omega, \delta^{\tilde{\nabla}}\eta), \qquad \omega \in A^r(E),\ \eta \in A^{r+1}(E),$$

which is of the form

$$\delta^{\tilde{\nabla}}\omega(X_1, \dots, X_r) := -\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\omega)(e_i, X_1, \dots, X_r) \tag{2.4}$$

for $\omega \in A^{r+1}(E)$, $X_1, \dots, X_r \in \mathfrak{X}(M)$. Here $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on $(M, g)$. For $X \in \mathfrak{X}(M)$,

$$A^r(E) \ni \omega \mapsto \tilde{\nabla}_X\omega \in A^r(E)$$

is also called the **covariant derviative** defined by

$$(\tilde{\nabla}_X \omega)(X_1, \dots, X_r) := \tilde{\nabla}_X(\omega(X_1, \cdots, X_r)) - \sum_{i=1}^{r} \omega(X_1, \dots, \nabla_X X_i, \dots, X_r). \tag{2.5}$$

Here the $\tilde{\nabla}$ in the first term in the above is the connection of $E$, and the $\nabla$ in the second term is the Levi-Civita connection of $(M, g)$. Then, for $\omega \in A^r(E)$, it follows that

$$(d^{\tilde{\nabla}} \omega)(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} (\tilde{\nabla}_{X_i} \omega)(X_1, \dots, \hat{X}_i, \dots, X_{r+1}). \tag{2.6}$$

For $r = 1$, we have

$$\begin{aligned} d^{\tilde{\nabla}} \omega(X, Y) &= \tilde{\nabla}_X(\omega(Y)) - \tilde{\nabla}_Y(\omega(X)) - \omega([X, Y]) \\ &= \{\tilde{\nabla}_X(\omega(Y)) - \omega(\nabla_X Y)\} - \{\tilde{\nabla}_Y(\omega(X)) - \omega(\nabla_Y X)\} \\ &= (\tilde{\nabla}_X \omega)(Y) - (\tilde{\nabla}_Y \omega)(X), \end{aligned}$$

by means of the definition of $d^{\tilde{\nabla}}$, the equation $[X, Y] = \nabla_X Y - \nabla_Y X$, and the definition of $\tilde{\nabla}\omega$.

DEFINITION (2.7). The differential operator

$$\Delta^{\tilde{\nabla}} := d^{\tilde{\nabla}} \delta^{\tilde{\nabla}} + \delta^{\tilde{\nabla}} d^{\tilde{\nabla}} : A^r(E) \to A^r(E)$$

is called the **Laplacian** of $E$-valued $r$-forms. By definition, we have

$$(\Delta^{\tilde{\nabla}} \omega, \eta) = (\omega, \Delta^{\tilde{\nabla}} \eta), \qquad \omega, \eta \in A^r(E).$$

The Laplacian $\Delta^{\tilde{\nabla}}$ is a second order elliptic differential operator. Let us define the **rough Laplacian** of $A^r(E)$ by

$$\bar{\Delta}\omega := -\sum_{i=1}^{m} (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i}) \omega, \qquad \omega \in A^r(E). \tag{2.8}$$

Then we obtain

PROPOSITION (2.9) (The Weitzenböck formula). *Let* $r = 1$. *Then*

$$\Delta^{\tilde{\nabla}} \omega = \bar{\Delta}\omega - \rho(\omega), \qquad \omega \in A^1(E). \tag{2.10}$$

*Here the operator* $\rho : A^1(E) \ni \omega \mapsto \rho(\omega) \in A^1(E)$ *is defined by*

$$\begin{aligned} \rho(\omega)(X) &:= \sum_{i=1}^{m} R^{\tilde{\nabla}}(X, e_i)(\omega(e_i)) - \sum_{i=1}^{m} \omega(R(X, e_i)e_i) \qquad (2.11) \\ &= \sum_{i=1}^{m} R^{\tilde{\nabla}}(X, e_i)(\omega(e_i)) - \omega(\rho(X)), \qquad X \in \mathfrak{X}(M). \end{aligned}$$

REMARK. For other $r \geq 2$, the Weitzenböck formula holds, but is rather complicated so we omit it. In the case $A^1(M)$, that is, for the trivial bundle $E = M \times \mathbb{C}$, we have

$$\rho(\omega)(X) = -\omega(\rho(X)), \qquad X \in \mathfrak{X}(M),$$

which yields (1.26), since the first term of (2.11) vanishes in this case because $\omega(e_i)$ is a locally defined function.

PROOF. Both sides of (2.10) belong to $A^1(E)$, so taking any point $x_0 \in M$ and an orthonormal frame $\{e_i\}_{i=1}^m$ on its neighborhood satisfying

$$(\nabla_Y e_k)(x_0) = 0, \qquad \forall Y \in T_{x_0}(M),\ 1 \leq k \leq m, \tag{2.12}$$

it suffices to show at $x_0$ that

$$\Delta^{\tilde{\nabla}} \omega(e_k) = \overline{\Delta}\omega(e_k) - \rho(\omega)(e_k).$$

The existence of such a frame field $\{e_i\}_{i=1}^m$ is shown as follows:

Take an orthonormal basis $\{u_i\}_{i=1}^m$ of $T_{x_0}M$ and a small neighborhood $U$ of $x_0$ in such a way that any point $x$ in $U$ can be connected by a unique geodesic, emanating $x_0$, denoted by $\gamma_x$. Then let $P_{\gamma_x}$ be the parallel transport along $\gamma_x$, and define $e_k$ by

$$e_k(x) := P_{\gamma_x} u_k \in T_x M, \qquad x \in U,$$

then $\{e_k\}_{k=1}^m$ is an orthonormal frame field on $U$, and moreover, letting $t \mapsto \gamma_Y(t)$ be a unique geodesic satisfying $\gamma_Y(0) = x_0$, $\gamma_Y'(0) = Y$ for any $Y \in T_{x_0}(M)$, by (3.10) in Chapter 2, we obtain

$$(\nabla_Y e_k)(x_0) = \left.\frac{d}{dt}\right|_{t=0} P_{\gamma_{Y,t}}{}^{-1} e_k(\gamma_Y(t)) = \left.\frac{d}{dt}\right|_{t=0} u_k = 0.$$

Now for $X = e_k$, it follows that, at $x_0 \in M$,

$$(d^{\tilde{\nabla}} \delta^{\tilde{\nabla}} \omega)(X) = \tilde{\nabla}_X(\delta^{\tilde{\nabla}} \omega) = -\sum_{i=1}^m \tilde{\nabla}_X(\tilde{\nabla}_{e_i}\omega(e_i)), \tag{2.13}$$

and

$$\begin{aligned}
(\delta^{\tilde{\nabla}} d^{\tilde{\nabla}} \omega)(X) &= -\sum_{i=1}^m \tilde{\nabla}_{e_i}(d^{\tilde{\nabla}}\omega)(e_i, X) \\
&= -\sum_{i=1}^m \{\tilde{\nabla}_{e_i}(d^{\tilde{\nabla}}\omega(e_i, X)) - d^{\tilde{\nabla}}\omega(\nabla_{e_i} e_i, X) - d^{\tilde{\nabla}}\omega(e_i, \nabla_{e_i} X)\} \\
&= -\sum_{i=1}^m \tilde{\nabla}(d^{\tilde{\nabla}}\omega(e_i, X)) \qquad\qquad (2.14) \\
&\qquad\qquad (\text{ the second, third terms vanish at } x_0 \text{ by (2.12)}) \\
&= -\sum_{i=1}^m \tilde{\nabla}_{e_i}\{(\tilde{\nabla}_{e_i}\omega)(X) - (\tilde{\nabla}_X\omega)(e_i)\},
\end{aligned}$$

by (2.6). Thus, we obtain at $x_0$,

$$(\Delta^{\tilde{\nabla}}\omega)(X) = -\sum_{i=1}^{m}\tilde{\nabla}_X(\tilde{\nabla}_{e_i}\omega(e_i)) - \sum_{i=1}^{m}\tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i}\omega)(X)) + \sum_{i=1}^{m}\tilde{\nabla}_{e_i}((\tilde{\nabla}_X\omega)(e_i)). \tag{2.15}$$

On the other hand, at $x_0$,

$$\overline{\Delta}\omega(X) = -\sum_{i=1}^{m}\tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i}\omega)(X)), \tag{2.16}$$

since $\nabla_{e_i}e_i = 0$ at $x_0$, and also

$$(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\omega)(X) = \tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i}\omega)(X) - (\tilde{\nabla}_{e_i}\omega)(\nabla_{e_i}X) = \tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i}\omega)(X)).$$

Moreover, at $x_0$, we obtain

$$\rho(\omega)(X) = \sum_{i=1}^{m}\{\tilde{\nabla}_X((\tilde{\nabla}_{e_i}\omega)(e_i)) - \tilde{\nabla}_{e_i}((\tilde{\nabla}\omega)(e_i))\}. \tag{2.17}$$

Because in general, for $X$, $Y$, $Z \in \mathfrak{X}(M)$, if we set

$$R^{\tilde{\nabla}}(X, Y)\omega := \tilde{\nabla}_X(\tilde{\nabla}_Y\omega) - \tilde{\nabla}_Y(\tilde{\nabla}_X\omega) - \tilde{\nabla}_{[X,Y]}\omega, \tag{2.18}$$

then we get

$$\begin{aligned}(R^{\tilde{\nabla}}(X, Y)\omega)(Z) &= \tilde{\nabla}_X((\tilde{\nabla}_Y\omega)(Z)) - (\tilde{\nabla}_Y\omega)(\nabla_X Z)\\ &\quad - \tilde{\nabla}_Y((\tilde{\nabla}_X\omega)(Z)) + (\tilde{\nabla}_X\omega)(\nabla_Y Z)\\ &\quad - (\tilde{\nabla}_{[X,Y]}\omega)(Z)\\ &= R^{\tilde{\nabla}}(X, Y)(\omega(Z)) - \omega(R(X, Y)Z).\end{aligned} \tag{2.19}$$

Thus, making use of (2.19), we obtain, at $x_0$,

$$\begin{aligned}\rho(\omega)(X) &= \sum_{i=1}^{m}(R^{\tilde{\nabla}}(X, e_i)\omega)(e_i)\\ &= \sum_{i=1}^{m}\{\tilde{\nabla}_X((\tilde{\nabla}_{e_i}\omega)(e_i)) - (\tilde{\nabla}_{e_i}\omega)(\nabla_X e_i) - \tilde{\nabla}_{e_i}((\tilde{\nabla}_X\omega)(e_i))\\ &\qquad + (\tilde{\nabla}_X\omega)(\nabla_{e_i}e_i) - (\tilde{\nabla}_{[e_i,X]}\omega)(e_i)\}\\ &= \text{ the right hand side of (2.17)},\end{aligned}$$

since all terms except the first and third terms vanish.

Therefore together with (2.15), (2.16) and (2.17), we obtain the desired equality. □

DEFINITION (2.20). For a $C^\infty$-mapping $\phi: (M, g) \to (N, h)$, define the induced vector bundle $E = \phi^{-1}TN$-valued 1-form $\omega := d\phi \in A^1(E)$ by

$$d\phi(X) := \phi_*(X) \in \Gamma(\phi^{-1}TN), \qquad X \in \mathfrak{X}(M)$$

(cf. (1.14′) (i) of Chapter 4).

Then we obtain

PROPOSITION (2.21). *For a $C^\infty$-mapping $\phi: (M, g) \to (N, h)$, a sufficient and necessary condition to be harmonic is that $\omega := d\phi \in A^1(E)$ in definition* (2.20) *is a harmonic form*, i.e.,

$$\Delta^{\tilde{\nabla}} \omega = 0,$$

*where $\Delta^{\tilde{\nabla}}$ is the Laplacian relative to the induced conection $\tilde{\nabla}$ on the induced bundle $E = \phi^{-1}TN$.*

PROOF. Note that $\Delta^{\tilde{\nabla}} = d^{\tilde{\nabla}}\delta^{\tilde{\nabla}} + \delta^{\tilde{\nabla}}d^{\tilde{\nabla}}$ and

$$(\Delta^{\tilde{\nabla}}\omega, \omega) = (d^{\tilde{\nabla}}\omega, d^{\tilde{\nabla}}\omega) + (\delta^{\tilde{\nabla}}\omega, \delta^{\tilde{\nabla}}\omega),$$

so we get

$$\Delta^{\tilde{\nabla}}\omega = 0 \iff d^{\tilde{\nabla}}\omega = 0 \quad \text{and} \quad \delta^{\tilde{\nabla}}\omega = 0.$$

The 1-form $\omega = d\phi$ always satisfies $d^{\tilde{\nabla}}\omega = 0$, because for $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} d^{\tilde{\nabla}}\omega(X, Y) &= \tilde{\nabla}_X(\omega(Y)) - \tilde{\nabla}_Y(\omega(X)) - \omega([X, Y]) \\ &= \tilde{\nabla}_X(\phi_* Y) - \tilde{\nabla}_Y(\phi_* X) - \phi_*([X, Y]) = 0 \end{aligned}$$

by Lemma (1.16) in Chapter 4. On the other hand, we obtain

$$\delta^{\tilde{\nabla}}\omega = -\tau(\phi), \tag{2.22}$$

because

$$\begin{aligned} \delta^{\tilde{\nabla}} &= -\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\omega)(e_i) = -\sum_{i=1}^{m}\{\tilde{\nabla}_{e_i}(\omega(e_i)) - \omega(\nabla_{e_i}e_i)\} \\ &= -\sum_{i=1}^{m}\{\tilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i}e_i\}. \end{aligned}$$

Therefore, we obtain Proposition (2.21). □

COROLLARY (2.23). *Let $\phi: (M, g) \to (N, h)$ be a harmonic mapping. Then $\omega = d\phi$ satisfies*

$$\bar{\Delta}\omega(X) - \sum_{i=1}^{m} {}^N R(\phi_* X, \phi_* e_i)\phi_* e_i + \phi_* \rho(X) = 0, \qquad X \in \mathfrak{X}(M).$$

PROOF. Noticing that $R^{\tilde{\nabla}}(X, e_i)\phi_* e_i = {}^N R(\phi_* X, \phi_* e_i)\phi_* e_i$, we see that the above equation follows from Proposition (2.21) and the Weitzenböck formula (2.9). □

**2.3. Proof of the instability theorem (2.1).** Before going into the proof, we prepare to show some facts about vector fields on the unit spher $S^m$. Let $(x_1, \ldots, x_{m+1})$ be the standard coordinate of $\mathbf{R}^{m+1}$, and $\langle x, y\rangle := \sum_{i=1}^{m+1} x_i y_i$ the standard inner product. Let $S^m$ be the unit sphere

$$S^m := \{x \in \mathbf{R}^{m+1} ;\ \langle x, x\rangle = 1\}.$$

For $x \in S^m$, according to the orthogonal decomposition

$$T_x\mathbf{R}^{m+1} = T_xS^m \oplus T_xS^{m\perp},$$

decompose any vector $V = \sum_{i=1}^{m+1} a_i (\frac{\partial}{\partial x_i})_x \in T_x\mathbf{R}^{m+1}$ as

$$V = V^{\mathsf{T}} + V^{\perp}, \qquad V^{\mathsf{T}} \in T_xS^m,\ V^{\perp} \in T_xS^{m\perp}.$$

Then we get

$$V^{\mathsf{T}} = \sum_{i=1}^{m+1}(a_i - x_i\langle a, x\rangle)\left(\frac{\partial}{\partial x_i}\right)_x, \qquad V^{\perp} = \langle a, x\rangle \sum_{i=1}^{m+1} x_i\left(\frac{\partial}{\partial x_i}\right)_x. \tag{2.24}$$

Thus, we obtain

$$T_xS^m = \left\{\sum_{i=1}^{m+1}(a_i - x_i\langle a, x\rangle)\left(\frac{\partial}{\partial x_i}\right)_x ;\ a \in \mathbf{R}^{m+1}\right\}. \tag{2.25}$$

For any $a \in \mathbf{R}^{m+1}$, the vector field $W = W_a$ on $S^m$ defined by

$$W(x) := \sum_{i=1}^{m+1}(a_j - x_j\langle a, x\rangle)\left(\frac{\partial}{\partial x_j}\right)_x, \qquad x \in S^m$$

satisfies

$$\nabla_X W = - < a, x > X, \qquad X \in T_xS^m, \tag{2.26}$$

where $\nabla$ is the Levi-Civita connection of $(S^m, g_{S^m})$.

PROOF. By Proposition (2.11) and (2.5) in Chapter 4

$$dW(X) = \nabla_X W + A(X, W)$$

for $X = \sum_{i=1}^{m+1} \xi_i (\frac{\partial}{\partial x_i})_x \in T_xS^m$. But the left hand side of this coincides with

$$\begin{aligned} dW(X) &= \sum_{i,j=1}^{m+1} \xi_i \frac{\partial}{\partial x_i}(a_j - x_j\langle a, x\rangle)\left(\frac{\partial}{\partial x_j}\right)_x \\ &= \sum_{i,j=1}^{m+1} \xi_i\{-\delta_{ij}\langle a, x\rangle - x_j a_i\}\left(\frac{\partial}{\partial x_j}\right)_x \\ &= -\langle a, x\rangle \sum_{i=1}^{m+1} \xi_i \left(\frac{\partial}{\partial x_i}\right)_x - \left(\sum_{i=1}^{m+1} a_i\xi_i\right)\sum_{j=1}^{m+1} x_j\left(\frac{\partial}{\partial x_j}\right)_x, \end{aligned}$$

where $\sum_{j=1}^{m+1} x_j (\frac{\partial}{\partial x_j})_x \in T_xS^{m\perp}$. Therefore, we obtain (2.26). □

Furthermore, let $\bar{\Delta}$ be the rough Laplacian acting on $A^1(S^m) \cong \mathfrak{X}(S^m)$ of the unit sphere $(S^m, g_{S^m})$, and let $W$ be as above. Then

$$\bar{\Delta} W = W. \tag{2.27}$$

**Proof.** Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on $(S^m, g_{S^m})$. Calculate

$$\bar{\Delta} W = -\sum_{i=1}^m (\nabla_{e_i}\nabla_{e_i} W - \nabla_{\nabla_{e_i} e_i} W).$$

Since $\nabla_{e_i} W = -(a, x) e_i$, putting $f := (a, x)$, we get

$$\bar{\Delta} W = -\sum_{i=1}^m \{\nabla_{e_i}(-f e_i) - f \nabla_{e_i} e_i\} = \sum_{i=1}^m (e_i f) e_i = W.$$

The last equality of the above follows from

$$\sum_{i=1}^m (e_i f) e_i = \operatorname{grad}_{S^m} f = (\operatorname{grad}_{\mathbf{R}^{m+1}} f)^{\mathrm{T}},$$

which is the $T_x S^m$-component of the gradient vector of $f$ in $\mathbf{R}^{m+1}$. Then

$$(\operatorname{grad}_{\mathbf{R}^{m+1}} f)^{\mathrm{T}} = \left(\sum_{i=1}^{m+1} a_i \frac{\partial}{\partial x_i}\right)^{\mathrm{T}} = W,$$

since (2.24) and the definition of $W$. □

Now for the above $W$, note that $\phi_* W \in \Gamma(\phi^{-1} TN)$. We shall prove that

$$\bar{\Delta} \phi_* W = \sum_{i=1}^m {}^N R(\phi_* W, \phi_* e_i) \phi_* e_i + (2 - m) \phi_* W, \tag{2.28}$$

where $\bar{\Delta}$ is the rough Laplacian of $\Gamma(\phi^{-1} TN)$. Then by (2.28), we obtain

$$\begin{aligned} \int_M h(J_\phi(\phi_* W), \phi_* W)\, v_g &= \int_M h\left(\bar{\Delta}\phi_* W - \sum_{i=1}^m {}^N R(\phi_* W, \phi_* e_i)\phi_* e_i, \phi_* W\right) v_g \\ &= (2 - m) \int_M h(\phi_* W, \phi_* W)\, v_g. \end{aligned}$$

Thus, if we assume $m \geq 3$ and index $(\phi) = 0$, then the left-hand side of the above should be nonnegative and, since $2 - m < 0$, it must be that

$$\int_M h(\phi_* W, \phi_* W)\, v_g = 0.$$

Therefore $\phi_* W = 0$. By (2.25) and the fact that $W$ corresponds to an arbitrary $a \in \mathbf{R}^{m+1}$, we obtain $\phi_* = 0$; that is, $\phi$ is a constant mapping which is Theorem (2.1).

**Proof of (2.28).** Let $x_0 \in S^m$ be an arbitrary point. We shall show that (2.28) holds at $x_0$. We take an orthonormal frame $\{e_i\}_{i=1}^m$ on a neighborhood of $x_0$ in such a way that

$$(\nabla_Y e_i)(x_0) = 0, \qquad \forall Y \in \mathfrak{X}(S^m),$$

as in the proof of Proposition (2.9). At $x_0$, we get

$$\begin{aligned}\bar{\Delta}\phi_* W &= -\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i}e_i})\phi_* W \\ &= -\sum_{i=1}^{m}\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\phi_* W.\end{aligned}$$

Here $\phi_* W = d\phi(W)$, and recall the definition of the covariant differentiation of $\phi^{-1}TN$-valued 1-form, $\tilde{\nabla}_{e_i} d\phi$. We obtain

$$\begin{aligned}\tilde{\nabla}_{e_i}\phi_* W &= (\tilde{\nabla}_{e_i} d\phi)(W) + d\phi(\nabla_{e_i} W) \\ &= (\tilde{\nabla}_{e_i} d\phi)(W) - d\phi(fe_i)\end{aligned}$$

by (2.26). Here $f(x) := (a, x)$, $x \in S^m$. Repeat this argument again, and then we get

$$\begin{aligned}\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\phi_* W &= \tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i} d\phi)(W) - d\phi(fe_i)) \\ &= (\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} d\phi)(W) + (\tilde{\nabla}_{e_i} d\phi)(\nabla_{e_i} W) - \tilde{\nabla}_{e_i}(d\phi(fe_i)).\end{aligned}$$

Thus, we obtain at $x_0$,

$$\begin{aligned}\bar{\Delta}\phi_* W = &-\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} d\phi)(W) - \sum_{i=1}^{m}(\tilde{\nabla}_{e_i} d\phi)(\nabla_{e_i} W) \\ &+\sum_{i=1}^{m}\tilde{\nabla}_{e_i}(d\phi(f e_i)).\end{aligned} \tag{2.29}$$

(i) Here at $x_0$, the third term of (3.29) $= \phi_* W$.

PROOF. Since $d\phi(fe_i) = \phi_*(fe_i) = f\phi_* e_i$, the third term of (2.29) coincides with

$$\sum_{i=1}^{m}\tilde{\nabla}_{e_i}(f\phi_* e_i) = \sum_{i=1}^{m}\{(e_i f)\phi_* e_i + f\tilde{\nabla}_{e_i}(\phi_* e_i)\}.$$

But at $x_0$,

$$\sum_{i=1}^{m}\tilde{\nabla}_{e_i}(\phi_* e_i) = \tau(\phi)(x_0) = 0,$$

since $\phi$ is harmonic. Moreover,

$$\phi_* W = \phi_*\left(\sum_{i=1}^{m} e_i(f) e_i\right) = \sum_{i=1}^{m} e_i(f)\,\phi_* e_i,$$

by the second equation in the proof of (2.27). □

(ii) The second term of (2.29) should vanish.

PROOF. We calculate

$$-\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}d\phi)(\nabla_{e_i}W)=\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}d\phi)(fe_i) \qquad \text{(by (2.26))}$$

$$=\sum_{i=1}^{m}\{\tilde{\nabla}_{e_i}(d\phi(fe_i))-\phi_*(\nabla_{e_i}(fe_i))\} \qquad \text{(definition of } \tilde{\nabla}_{e_i}d\phi)$$

$$=\sum_{i=1}^{m}\{\tilde{\nabla}_{e_i}(f\phi_*e_i)-\phi_*((e_if)e_i)-\phi_*(f\nabla_{e_i}e_i)\}$$

$$=\sum_{i=1}^{m}\{(e_if)\phi_*e_i+f\tilde{\nabla}_{e_i}\phi_*e_i-(e_if)\phi_*e_i-f\phi_*\nabla_{e_i}e_i\}$$

$$=f\,\tau(\phi)(x_0)=0. \quad \square$$

(iii) The first term of (2.29) $=\sum_{i=1}^{m}{}^{N}R(\phi_*W, \phi_*e_i)\phi_*e_i-(m-1)\phi_*W$.

PROOF. Since $\nabla_{e_i}e_i=0$ at $x_0$, the first term of (2.29) is equal to

$$(\overline{\Delta}d\phi)(W)=\sum_{i=1}^{m}{}^{N}R(\phi^*W, \phi_*e_i)\phi_*e_i-\phi_*\rho(W),$$

by Corollary (2.23). Here by (2.17) in Chapter 4, we get

$$\rho(W)=\sum_{i=1}^{m}R(W, e_i)e_i=(m-1)W.$$

Thus, we obtain (iii). $\square$

Together with (i), (ii), and (iii), we obtain (2.28) and this completes the proof of main theorem (2.1). $\square$

## §3. Stability of holomorphic mappings

**3.1. Main theorems.** In this section, we shall show the weak stability of holomorphic mappings between compact Kähler manifolds and give applications.

Here for a complex manifold $M$ with a Riemannian metric $g$, $(M, g)$ is a Kähler manifold, or $g$ is a Kähler metric with $J$ the almost complex structure and with $\nabla$ the Levi-Civita connection of $g$, we have that

$$g(JX, JY)=g(X, Y), \quad \nabla_X(JY)=J(\nabla_XY), \quad X, Y\in\mathfrak{X}(M).$$

Lichenerowicz proved the following theorem ( see [E.L 2]) in 1970:

THEOREM (3.1). *Let $(M, g)$, $(N, h)$ be two compact Kähler manifolds, let $\phi\colon M\to N$ be a holomorphic mapping,* i.e., *$\phi_*\circ J=J\circ\phi_*$. Then*

(i) (**Energy minimizing**) *The holomorphic mapping $\phi$ minimizes $E$ in its homotopy class. That is, if $\psi\in C^\infty(M, N)$ is homotopic to $\phi$, i.e., there exists a $C^1$-mapping $F: [0, 1]\times M\to N$ satisfying*

$$F(0, x)=\phi(x), \qquad F(1, x)=\psi(x), \qquad x\in M,$$

*then*

$$E(\phi) \leq E(\psi).$$

(ii) (**Rigidity**) *Moreover, if a variation of* $C^1$*-mappings is given by harmonic mappings, that is,* $\phi_t : M \to N$ *with* $\phi_0 = \phi$, $-\epsilon < t < \epsilon$ *is harmonic of* $(M, g)$ *into* $(N, h)$, *for any* $t$. *Then* $\phi_t$ *are all holomorphic mappings of* $M$ *into* $N$.

In this section, we shall show the following theorem, which is regarded as an infinitesimal version of the above theorem (3.1):

THEOREM (3.2). *Let* $(M, g)$, $(N, h)$ *be two compact Kähler manifolds and let* $\phi : M \to N$ *be a holomorphic mapping. Then the following equality holds*:

$$\int_M h(J_\phi V, V)\, v_g = \frac{1}{2}\int_M h(DV, DV)\, v_g \geq 0, \qquad V \in \Gamma(\phi^{-1}TN), \quad (3.3)$$

*where* $J_\phi$ *is the Jacobi operator of* $\phi$ *regarded as a harmonic mapping of* $(M, g)$ *into* $(N, h)$. *For each* $V \in \Gamma(\phi^{-1}TN)$, $DV$ *is an element of* $\Gamma(\phi^{-1}TN \otimes T^*M)$ *defined by*

$$DV(X) := \tilde{\nabla}_{JX}V - J\tilde{\nabla}_X V, \qquad X \in \mathfrak{X}(M).$$

*Then, in particular,*

(i) $\phi$ *is weakly stable, that is, each eigenvalue of* $J_\phi$ *is nonnegative.*

(ii) $\operatorname{Ker}(J_\phi) = \{V \in \Gamma(\phi^{-1}TN);\ DV = 0\}$.

REMARK. An equality similar to (3.3) was obtained on page 164, Lemma 4 in [Su]. See also [No] for recent topics about the moduli space of holomorphic mappings.

**3.2. Analytic vector fields along holomorphic mappings.** We call $V \in \Gamma(\phi^{-1}TN)$ satisfying $DV = 0$ in Theorem (3.2), **analytic vector fields** along $\phi : M \to N$, their totality is denoted by $\mathfrak{a}(\phi^{-1}TN)$. In this subsection, we shall explain the meanings of the analytic vector fields as follows.

(3.4) *Holomorphic vector bundle.* A $C^\infty$-vector bundle $E$ over a complex manifold $M$ is a **complex vector bundle** if $E$ further satisfies that

(i) each fiber $E_p$, $p \in M$ is a complex vector space of constant dimension, say $r$-dimension.

(ii) For each $p_0 \in M$, there exist a neighborhood $U$ and a $C^\infty$-diffeomorphism $\psi$ of $U \times \mathbf{C}^r$ onto $\psi^{-1}(U)$ such that

$$\pi(\psi(p, v)) = p, \qquad p \in U,\ v \in \mathbf{C}^r,$$

and the mapping $\mathbf{C}^r \ni v \mapsto \psi(p, v) \in E_p$ is a complex linear isomorphism.

Furthermore, a complex vector bundle $E$ is a **holomorphic vector bundle** if $E$ itself a complex manifold and $\pi : E \to M$ is holomorphic and the

$C^\infty$-diffeomorphism in (ii), $\psi : U \times \mathbf{C}^r \to \pi^{-1}(U)$ is a holomorphic diffeomorphism ( i.e., $\psi$ and $\psi^{-1}$ are holomorphic).

A $C^\infty$-section $s$ of a holomorphic vector bundle $E$ is a **holomorphic section** if $s : M \to E$ is holomorphic. The totality of all holomorphic sections of $E$ is denoted by $\Omega^0(E)$.

(3.5) *Holomorphic tangent bundle.* For each point $p \in M$ of a complex manifold $M$, the complexification of the tangent space $T_pM$ is denoted by $T_p{}^{\mathbf{C}}M = T_pM \otimes \mathbf{C}$. Then the almost complex structure of $M$, $J : T_pM \to T_pM$, can be uniquely extended to a complex linear mapping of the complexification, denoted by $J : T_p{}^{\mathbf{C}}M \to T_p{}^{\mathbf{C}}M$, and the eigenvalue of $J$ is $\pm\sqrt{-1}$ since $J^2 = -\mathrm{id}$. Therefore, $T_p^{\mathbf{C}}M$ is decomposed into a direct sum as

$$T_pM^{\mathbf{C}}M = T_p'M \oplus T_p''M,$$

where

$$T_p'M := \{v \in T_p^{\mathbf{C}}M;\ Jv = \sqrt{-1}v\},$$
$$T_p''M := \{v \in T_p^{\mathbf{C}}M;\ Jv = -\sqrt{-1}v\}.$$

Then it turns out that

$$T'M := \bigcup_{p\in M} T_p'M$$

is a complex vector bundle, and moreover, a holomorphic vector bundle which is called the **holomorphic tangent bundle.**

The tangent bundle is often identified with the holomorphic tangent bundle $T'M$ via a linear isomorphism

$$T_pM \ni X \mapsto \tilde{X} := \frac{1}{2}(X - \sqrt{-1}JX) \in T_p'M. \tag{3.6}$$

The holomorphic sections of $T'M$ are called the **holomorphic vector fields** on $M$.

Taking a complex coordinate $(z_1, \dots, z_m)$, $m = \dim_{\mathbf{C}} M$, on a neighborhood $U$ in a complex manifold $M$, put

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right),$$

then at each point $p \in U$,

$$\left\{\left(\frac{\partial}{\partial z_1}\right)_p, \dots, \left(\frac{\partial}{\partial z_m}\right)_p\right\}, \qquad \left\{\left(\frac{\partial}{\partial \overline{z}_1}\right)_p, \dots, \left(\frac{\partial}{\partial \overline{z}_m}\right)_p\right\}$$

are bases of $T_p'M$, $T_p''M$, respectively. For a vector field

$$Z = \sum_{j=1}^m f_j \frac{\partial}{\partial z_j}, \qquad f_j \in C^\infty(U),\ 1 \le j \le m,$$

a sufficient and necessary condition to be holomorphic vector fields is that all the $f_j$ are holomorphic functions of $(z_1, \dots, z_m)$. The correspondence of (3.6) is

$$\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}, \qquad \frac{\partial}{\partial y_j} \mapsto \sqrt{-1}\frac{\partial}{\partial z_j}.$$

(3.6′) *The induced holomorphic tangent bundle.* For a homorphic mapping $\phi : M \to N$ between two complex manifolds $M$, $N$, and a holomorphic vector bundle $E$ over $N$, then the induced bundle $\phi^{-1}T'N$ of $E$ by $\phi$ becomes a holomorphic vector bundle over $M$. In particular, the inducd bundle of the holomorphic tangent bundle $T'N$ by $\phi$ is a holomorphic vector bundle over $M$, called the **induced holomorphic tangent bundle.** We denote the totality of all holomorphic sections of $\phi^{-1}T'N$ by $\Omega^0(\phi^{-1}T'N)$ whose elements are called the **holomorphic vector fields along** $\phi$.

PROPOSITION (3.7). *Let $(M, g)$, $(N, h)$ be two Kähler manifolds, and let $\phi : M \mapsto N$ be a holomorphic mapping between them. There there exists the following isomorphism:*

$$\mathfrak{a}(\phi^{-1}TN) \cong \Omega^0(\phi^{-1}T'N).$$

*The correspondance is given by*

$$V \mapsto \tilde{V} := \frac{1}{2}(V - \sqrt{-1}JV), \tag{3.8}$$

*where $J$ is the almost complex structure of $N$ and for $V \in \Gamma(\phi^{-1}TN)$, $JV \in \Gamma(\phi^{-1}TN)$ is defined by*

$$JV(p) := J(V(p)), \qquad p \in M,$$

*since $J$ sends $T_{\phi(p)}N$ into itself.*

Due to Theorem (3.2) and Proposition (3.7), we obtain

COROLLARY (3.9). *Let $(M, g)$, $(N, h)$ be two compact Kähler manifolds, and let $\phi : M \to N$ be a holomorphic mapping. Then we obtain*

$$\operatorname{Ker}(J_\phi) = \mathfrak{a}(\phi^{-1}TN) \cong \Omega^0(\phi^{-1}T'N),$$

*and then the nullity of $\phi$ is*

$$\operatorname{nullity}(\phi) = \dim_{\mathbf{R}} \mathfrak{a}(\phi^{-1}TN) = \dim_{\mathbf{C}} \Omega^0(\phi^{-1}T'N).$$

COROLLARY (3.10). *The identity mapping of a compact Kähler manifold $(M, g)$, $\mathrm{id} : (M, g) \to (M, g)$ is always weakly stable and*

$$\operatorname{Ker}(J_{\mathrm{id}}) \cong \mathfrak{a}(M),$$

*where $\mathfrak{a}(M)$ is the space of all holomorphic vector fields on $M$.*

(3.11) PROOF OF PROPOSITION (3.7). We first prove Proposition (3.7). By (3.6), we get a real linear isomorphism

$$T_{\phi(p)}N \ni V(p) \mapsto \tilde{V}(p) = \frac{1}{2}(V(p) - \sqrt{-1}J(V(p))) \in T'_{\phi(p)}N$$

for all $p \in N$. So by (3.8), we obtain an isomorphism

$$\Gamma(\phi^{-1}TN) \ni V \mapsto \tilde{V} \in \Gamma(\phi^{-1}T'N).$$

Thus, it suffices to show that

$$DV = 0 \Longleftrightarrow \tilde{V} \text{ is a holomorphic section.} \tag{3.12}$$

Indeed, we see (3.12) as follows: We take the local complex coordinates around $p \in M$, $\phi(p) \in N$ as $(z_1, \dots, z_m)$, $(w_1, \dots, w_n)$ with $\dim_{\mathbf{C}} M = m$ and $\dim_{\mathbf{C}} N = n$, and let

$$z_j = x_j + \sqrt{-1}y_j, \qquad w_k = u_k + \sqrt{-1}v_k,$$

for $1 \le j \le m$, $1 \le k \le n$, then each $V \in \Gamma(\phi^{-1}TN)$ can be written locally as

$$V(x) = \sum_{k=1}^{n} \left\{ \xi_k(x) \left(\frac{\partial}{\partial u_k}\right)_{\phi(x)} + \eta_k(x) \left(\frac{\partial}{\partial v_k}\right)_{\phi(x)} \right\}$$

for each point $x$ in a coordinate neighborhood $U$ of $p$. Here $\xi_k$, $\eta_k$ are $C^\infty$-functions on $U$. Then we shall show for $V$ that a necessary and sufficient condition to be in $\mathfrak{a}(\phi^{-1}TN)$ is that each $\xi_k$, $\eta_k$ satisfies the **Cauchy-Riemann equations**

$$\frac{\partial \xi_k}{\partial x_j} = \frac{\partial \eta_k}{\partial y_j}, \qquad \frac{\partial \xi_k}{\partial y_j} = -\frac{\partial \eta_k}{\partial x_j}. \tag{3.13}$$

for all $1 \le k \le n$, $1 \le j \le m$. Then since for

$$\tilde{V} = \sum_{k=1}^{n} (\xi_k + \sqrt{-1}\eta_k) \frac{\partial}{\partial w_k},$$

(3.13) is equivalent to the condition that each $\xi_k + \sqrt{-1}\eta_k$ is a holomorphic function of $(z_1, \dots, z_n)$, we obtain (3.12).

LEMMA (3.14). *Let $(M, g)$, $(N, h)$ be Kähler manifolds, and let $\phi : (M, g) \to (N, h)$ be a holomorphic mapping. We take the complex coordinates in $M$, $N$, as $(z_1, \dots, z_m)$, $(w_1, \dots, w_n)$ with $z_j = x_j + \sqrt{-1}y_j$, $w_k = u_k + \sqrt{-1}v_k$, respectively. Then, for $Y = \frac{\partial}{\partial u_k}$, or $\frac{\partial}{\partial v_k}$, we have*

$$J\tilde{\nabla}_{\frac{\partial}{\partial x_j}} Y = \tilde{\nabla}_{\frac{\partial}{\partial y_j}} Y$$

*for all $1 \le j \le m$.*

PROOF. For $Y = \frac{\partial}{\partial u_k}$, we show

$$J^N\nabla_{\phi_* \frac{\partial}{\partial x_j}} \frac{\partial}{\partial u_k} - {}^N\nabla_{\phi_* \frac{\partial}{\partial y_j}} \frac{\partial}{\partial u_k} = 0. \tag{3.15}$$

A similar argument holds for $\frac{\partial}{\partial v_k}$. Since

$$\phi_* \frac{\partial}{\partial x_j} = \sum_{\ell=1}^{n} \left\{ \frac{\partial(u_\ell \circ \phi)}{\partial x_j} \frac{\partial}{\partial u_\ell} + \frac{\partial(v_\ell \circ \phi)}{\partial x_j} \frac{\partial}{\partial v_\ell} \right\},$$

$$\phi_* \frac{\partial}{\partial y_j} = \sum_{\ell=1}^{n} \left\{ \frac{\partial(u_\ell \circ \phi)}{\partial y_j} \frac{\partial}{\partial u_\ell} + \frac{\partial(v_\ell \circ \phi)}{\partial y_j} \frac{\partial}{\partial v_\ell} \right\},$$

the left hand side of (3.15) coincides with

$$-\sum_{\ell=1}^{n} \left\{ \frac{\partial(u_\ell \circ \phi)}{\partial y_j} {}^N\nabla_{\frac{\partial}{\partial u_\ell}} \frac{\partial}{\partial u_k} + \frac{\partial(v_\ell \circ \phi)}{\partial y_j} {}^N\nabla_{\frac{\partial}{\partial v_\ell}} \frac{\partial}{\partial u_k} \right\},$$

$$+\sum_{\ell=1}^{n} \left\{ \frac{\partial(u_\ell \circ \phi)}{\partial x_j} J\, {}^N\nabla_{\frac{\partial}{\partial u_\ell}} \frac{\partial}{\partial u_k} + \frac{\partial(v_\ell \circ \phi)}{\partial x_j} J\, {}^N\nabla_{\frac{\partial}{\partial v_\ell}} \frac{\partial}{\partial u_k} \right\}.$$

Here note that $J\,{}^N\nabla_X Y = {}^N\nabla_X(JY)$ for $X, Y \in \mathfrak{X}(N)$ since $(N, h)$ is Kähler. Thus, we get

$$\begin{cases} J\,{}^N\nabla_{\frac{\partial}{\partial u_\ell}} \frac{\partial}{\partial u_k} = J\,{}^N\nabla_{\frac{\partial}{\partial u_k}} \frac{\partial}{\partial u_\ell} = {}^N\nabla_{\frac{\partial}{\partial u_k}} \frac{\partial}{\partial v_\ell} = {}^N\nabla_{\frac{\partial}{\partial v_\ell}} \frac{\partial}{\partial u_k}, \\ J\,{}^N\nabla_{\frac{\partial}{\partial v_\ell}} \frac{\partial}{\partial u_k} = J\,{}^N\nabla_{\frac{\partial}{\partial u_k}} \frac{\partial}{\partial v_\ell} = -{}^N\nabla_{\frac{\partial}{\partial u_k}} \frac{\partial}{\partial u_\ell} = -{}^N\nabla_{\frac{\partial}{\partial u_\ell}} \frac{\partial}{\partial u_k}. \end{cases}$$

Substituting these into the above, we find that the above equation coincides with

$$\sum_{\ell=1}^{n} \left\{ \frac{\partial(u_\ell \circ \phi)}{\partial x_j} - \frac{\partial(v_\ell \circ \phi)}{\partial y_j} \right\} {}^N\nabla_{\frac{\partial}{\partial v_\ell}} \frac{\partial}{\partial u_k}$$

$$-\sum_{\ell=1}^{n} \left\{ \frac{\partial(v_\ell \circ \phi)}{\partial x_j} + \frac{\partial(u_\ell \circ \phi)}{\partial y_j} \right\} {}^N\nabla_{\frac{\partial}{\partial u_\ell}} \frac{\partial}{\partial u_k},$$

which vanishes since $\phi$ is holomorphic (cf. (3.12) in Chapter 4). □

(Proof of (3.12) continued.) Recall that

$$V \in \mathfrak{a}(\phi^{-1}TN) \iff DV = 0 \iff \tilde{\nabla}_{JX} V - J\tilde{\nabla}_X V = 0, \qquad X \in \mathfrak{X}(M).$$

We show this is equivalent to (3.13), if we denote locally

$$V = \sum_{i=1}^{n} \left( \xi_k \frac{\partial}{\partial u_k} + \eta_k \frac{\partial}{\partial v_k} \right).$$

Since

$$\tilde{\nabla}_{J(fX)} V = f \tilde{\nabla}_{JX} V,$$

for all $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, it suffices to show the above only for

$X = \frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$, $1 \le j \le m$. We calculate for $X = \frac{\partial}{\partial x_j}$,

$$\begin{aligned}
\tilde{\nabla}_{JX}V - J\tilde{\nabla}_X V &= \sum_{k=1}^{n}\left\{\frac{\partial \xi_k}{\partial y_j}\frac{\partial}{\partial u_k} + \xi_k \tilde{\nabla}_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial u_k} + \frac{\partial \eta_k}{\partial y_j}\frac{\partial}{\partial v_k} + \eta_k \tilde{\nabla}_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial v_k}\right\} \\
&\quad - \sum_{k=1}^{n} J\left\{\frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial u_k} + \xi_k \tilde{\nabla}_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial u_k} + \frac{\partial \eta_k}{\partial x_j}\frac{\partial}{\partial v_k} + \eta_k \tilde{\nabla}_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial v_k}\right\} \\
= \sum_{k=1}^{n}&\left\{\left(\frac{\partial \xi_k}{\partial y_j} + \frac{\partial \eta_k}{\partial x_j}\right)\frac{\partial}{\partial u_k} + \left(\frac{\partial \eta_k}{\partial y_j} - \frac{\partial \xi_k}{\partial x_j}\right)\frac{\partial}{\partial v_k}\right\} \\
+ \sum_{k=1}^{n}&\left\{\xi_k\left(\tilde{\nabla}_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial u_k} - J\tilde{\nabla}_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial u_k}\right) + \eta_k\left(\tilde{\nabla}_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial v_k} - J\tilde{\nabla}_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial v_k}\right)\right\}.
\end{aligned}$$

Here the second term of the last vanishes by Lemma (3.14). We get a similar formula for $X = \frac{\partial}{\partial y_j}$. Thus, we obtain the desired result. □

(3.16) *Geometric meaning of Corollary* (3.9). In order to consider the meaning of Corollary (3.9), given two compact Kähler manifolds $(M, g)$, $(N, h)$, we put

$\mathfrak{Har}(M, N) :=$ the totality of all harmonic mappings of $(M, g)$ into $(N, h)$,

$\mathfrak{Hol}(M, N) :=$ the totality of all holomorphic mappings of $M$ into $N$.

Recall by Example 6, (3.12) in Chapter 4, we get

$$\mathfrak{Hol}(M, N) \subset \mathfrak{Har}(M, N). \tag{3.17}$$

If we regard $\mathfrak{Hol}(M, N)$ as a "(real) submanifold"of $\mathfrak{Har}(M, N)$, then the "tangent space" $T_\phi\mathfrak{Hol}(M, N)$ of $\mathfrak{Hol}(M, N)$ at $\phi \in \mathfrak{Hol}(M, N)$ satisfies

$$T_\phi\mathfrak{Hol}(M, N) \subset \mathfrak{a}(\phi^{-1}TN). \tag{3.18}$$

PROOF. Take a variation $\phi_t$ of $\phi \in \mathfrak{Hol}(M, N)$ in $\mathfrak{Hol}(M, N)$, i.e., $\phi_t \in \mathfrak{Hol}(M, N)$, $-\epsilon < t < \epsilon$, with $\phi_0 = \phi$. Then we can show the variation vector field

$$V(p) := \left.\frac{d}{dt}\right|_{t=0}\phi_t(p), \qquad p \in M$$

belongs to $\mathfrak{a}(\phi^{-1}TN)$. Indeed, take the local complex coordinates $(z_1, \dots, z_m)$, $(w_1, \dots, w_n)$ around $p$, $\phi(p)$, respectively, and put $z_j = x_j + \sqrt{-1}y_j$, $w_k = u_k + \sqrt{-1}v_k$. Since $\phi_t$ are holomorphic, $u_k \circ \phi_t$ and $v_k \circ \phi_t$ satisfy the Cauchy-Riemann equation:

$$\frac{\partial}{\partial x_j}(u_k \circ \phi_t) = \frac{\partial}{\partial y_j}(v_k \circ \phi_t), \qquad \frac{\partial}{\partial y_j}(u_k \circ \phi_t) = -\frac{\partial}{\partial x_j}(v_k \circ \phi_t),$$

for $1 \le j \le m$, $1 \le k \le n$. Differentiating these in $t$ at $t = 0$ and putting

$$\xi_k := \left.\frac{d}{dt}\right|_{t=0} u_k \circ \phi_t, \qquad \eta_k := \left.\frac{d}{dt}\right|_{t=0} v_k \circ \phi_t,$$

we get

$$\frac{\partial \xi_k}{\partial x_j} = \frac{\partial \eta_k}{\partial y_j}, \qquad \frac{\partial \xi_k}{\partial y_j} = -\frac{\partial \eta_k}{\partial x_j}$$

which implies, by the above argument, that

$$V = \sum_{k=1}^{n} \left( \xi_k \frac{\partial}{\partial u_k} + \eta_k \frac{\partial}{\partial v_k} \right)$$

belongs to $\mathfrak{a}(\phi^{-1}TN)$. □

Thus, for $\phi \in \mathfrak{Hol}(M, N)$ we obtain the following inclusion relations:

$$\begin{array}{c} T_\phi \mathfrak{Har}(M, N) \subset \mathrm{Ker}(J_\phi) \\ \cup \\ T_\phi \mathfrak{Hol}(M, N) \subset \mathfrak{a}(\phi^{-1}TN) \end{array}$$

Corollary (3.9) implies that Ker $(J_\phi) = \mathfrak{a}(\phi^{-1}TN)$. So, roughly speaking, we get

$$T_\phi \mathfrak{Har}(M, N) \doteqdot T_\phi \mathfrak{Hol}(M, N).$$

This is the reason for calling that Theorem (3.2) and Corollary (3.9) are the infinitesimal version of Theorem (3.1). To study the structures of their tangent spaces are much easier than to study the ones of $\mathfrak{Hol}(M, N)$ and $\mathfrak{Har}(M, N)$ themselves. In fact, it suffices to prove Ker $(J_\phi) = \{0\}$ to show the rigidity of $\phi$, if $(N, h)$ is negative curvature (see [Su]).

**3.3. Proof of Main Theorem (3.2).** As a proof of Proposition (3.14) in Chapter 4, we take a local orthonormal frame $\{e_i, \dots, e_m, f_1, \dots, f_m\}$ such that $Je_i = f_i$, $Jf_i = -e_i$, $1 \le i \le m$.

By (1.7), (1.13) in Chapter 5, for $V \in \Gamma(\phi^{-1}TN)$, we get

$$\begin{aligned} &\int_M h(J_\phi V, V)\, v_g \\ &= \int_M \sum_{i=1}^{m} \{ h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{e_i} V) + h(\tilde{\nabla}_{f_i} V, \tilde{\nabla}_{f_i} V) \\ &\qquad - h({}^N R(V, \phi_* e_i)\phi_* e_i, V) - h({}^N R(V, \phi_* f_i)\phi_* f_i, V)\}\, v_g. \end{aligned} \tag{3.19}$$

Here we show

LEMMA (3.20). *We obtain the formula that*

$$R(V, \phi_* e_i)\phi_* e_i + {}^N R(V, \phi_* f_i)\phi_* f_i = J\, {}^N R(\phi_* e_i, \phi_* f_i) V.$$

PROOF. Since $\phi : M \to N$ is holomorphic and since $(N, h)$ is Kähler, the left-hand side of the above is equal to

$$\begin{aligned} &-J\, {}^N R(V, \phi_* e_i)\phi_* f_i + J\, {}^N R(V, \phi_* f_i)\phi_* e_i \\ &\quad = J\, {}^N R(\phi_* e_i, V)\phi_* f_i + J\, {}^N R(V, \phi_* f_i)\phi_* e_i \\ &\quad = J\, {}^N R(\phi_* e_i, \phi_* f_i) V, \end{aligned}$$

where we used the formulas, for $X, Y, Z \in \mathfrak{X}(M)$,

$$^{N}R(X, Y) + {}^{N}R(Y, X) = 0,$$
$$^{N}R(X, Y)Z + {}^{N}R(Y, Z)X + {}^{N}R(Z, X)Y = 0. \qquad \square$$

PROOF OF THEOREM (3.2) CONTINUED. We calculate that

$$\begin{aligned} h(DV, DV) &= \sum_{i=1}^{m}\{h(DV(e_i), DV(e_i)) + h(DV(f_i), DV(f_i))\} \\ &= \sum_{i=1}^{m}\{h(\tilde{\nabla}_{Je_i}V - J\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{Je_i}V - J\tilde{\nabla}_{e_i}V) \\ &\quad + h(\tilde{\nabla}_{Jf_i}V - J\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{Jf_i}V - J\tilde{\nabla}_{f_i}V)\}. \end{aligned}$$

Here using $Je_i = f_i$, $Jf_i = -e_i$, and $h(JX, JY) = h(X, Y)$, we obtain

$$h(DV, DV) = 2\sum_{i=1}^{m}\{h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{e_i}V) - 2h(J\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}V) + h(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{f_i}V)\}.$$

Thus the integrand of (3.19) minus $-\frac{1}{2}h(DV, DV)$ coincides with the following:

$$\begin{aligned} &\sum_{i=1}^{m}\{-h({}^{N}R(V, \phi_* e_i)\phi_* e_i, V) - h({}^{N}R(V, \phi_* f_i)\phi_* f_i, V) \qquad (3.21) \\ &\qquad\qquad + 2h(J\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}V)\} \\ &= \sum_{i=1}^{m}\{-h(J\,{}^{N}R(\phi_* e_i, \phi_* f_i)V, V) + 2h(J\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}V)\}, \end{aligned}$$

By Lemma (3.20). The resulting equation is a $C^\infty$-function on $M$. To complete the proof, it only suffices to show the integral of this function over $M$ vanishes. To do this, we show

$$\begin{aligned} &\int_M -\sum_{i=1}^{m} h(J\,{}^{N}R(\phi_* e_i, \phi_* f_i)V, V)\, v_g \qquad (3.22) \\ &= \int_M \sum_{i=1}^{m}\{h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}JV) - h(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{e_i}JV)\}\, v_g. \end{aligned}$$

Then the integral of (3.21) over $M$ coincides with

$$\int_M \sum_{i=1}^{m}\{h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}JV) - h(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{e_i}JV) + 2h(J\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}V)\}\, v_g,$$

which vanishes because $\tilde{\nabla}_X JV = J\tilde{\nabla}_X V$ since $(N, h)$ is Kähler, and $h(JX, JY) = h(X, Y)$. Thus, we obtain the desired result.

Equation (3.22) can be derived as follows: Since $[e_i, f_i] = \nabla_{e_i} f_i - \nabla_{f_i} e_i$, we obtain

$$\begin{aligned} &-h(J\,{}^N R(\phi_* e_i, \phi_* f_i)V, V) \\ &\quad = h({}^N R(\phi_* e_i, \phi_* f_i)V, JV) \\ &\quad = h(\tilde{\nabla}_{e_i}\tilde{\nabla}_{f_i}V - \tilde{\nabla}_{f_i}\tilde{\nabla}_{e_i}V - \tilde{\nabla}_{[e_i, f_i]}V, JV) \\ &\quad = e_i \cdot h(\tilde{\nabla}_{f_i}V, JV) - \underline{h(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{e_i}JV)} - f_i \cdot h(\tilde{\nabla}_{e_i}V, JV) \\ &\quad\quad + \underline{h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}JV)} - h(\tilde{\nabla}_{\nabla_{e_i}f_i}V, JV) + h(\tilde{\nabla}_{\nabla_{f_i}e_i}V, JV), \end{aligned}$$

where we used the compatibility of $\tilde{\nabla}$ and $h$. Therefore, we define a $C^\infty$-function $\varphi$ on $M$ by

$$\varphi := \sum_{i=1}^m \Big\{ e_i \cdot h(\tilde{\nabla}_{f_i}V, JV) - f_i \cdot h(\tilde{\nabla}_{e_i}V, JV) - h(\tilde{\nabla}_{\nabla_{e_i}f_i}V, JV) + h(\tilde{\nabla}_{\nabla_{f_i}e_i}V, JV) \Big\}.$$

Then we obtain (3.22) if we show $\int_M \varphi\, v_g = 0$.

For this, we define $X \in \mathfrak{X}(M)$ by

$$g(X, Y) = h(\tilde{\nabla}_{JY}V, JV) \qquad \forall Y \in \mathfrak{X}(M).$$

By Green's formula (cf. (iii) of Proposition (3.29) in Chapter 2), we only have to prove $\mathrm{div}(X) = \varphi$, which follows from the equation

$$\begin{aligned} \mathrm{div}(X) &= \sum_{i=1}^m \{ g(e_i, \nabla_{e_i}X) + g(f_i, \nabla_{f_i}X) \} \\ &= \sum_{i=1}^m \{ e_i \cdot g(e_i, X) - g(\nabla_{e_i}e_i, X) + f_i \cdot g(f_i, X) - g(\nabla_{f_i}f_i, X) \} \\ &= \sum_{i=1}^m \{ e_i \cdot h(\tilde{\nabla}_{Je_i}V, JV) - h(\tilde{\nabla}_{J\nabla_{e_i}e_i}V, JV) \\ &\quad + f_i \cdot h(\tilde{\nabla}_{Jf_i}V, JV) - h(\tilde{\nabla}_{J\nabla_{f_i}f_i}V, JV) \} \\ &= \varphi. \end{aligned}$$

Thus we have proved the main theorem (3.2). □

**3.4. Applications.** We prove the following theorem as an application of Corollary (3.10).

THEOREM (3.23) (M. Obata, 1965). *Let $(M, g)$ be a compact Kähler manifold on which the Ricci operator has the property that for all $x \in M$,*

$$g(\rho(u), u) \geq \alpha\, g(u, u), \qquad \forall u \in T_x M,$$

*for a positive constant $\alpha$. Then the first eigenvalue of the Laplacian $\Delta_g$ acting on $C^\infty(M)$, $\lambda_1(g)$ (cf. Remark below Proposition (1.20)), satisfies*

$$\lambda_1(g) \geq 2\alpha.$$

*If the equality holds, then $M$ admits a nonzero holomorphic vector field.*

PROOF. In order to apply (1.27) $J_{\mathrm{id}} = \Delta_H - 2\rho$, we prepare with the following lemma.

LEMMA (3.24). *Let $(M, g)$ be a compact Riemannian manifold. Let the smallest eigenvalue of the operator $\Delta_H : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be $\lambda_1^1(g)$. If the identity mapping of $(M, g)$ is weakly stable, then the following inequality holds:*

$$2 \inf \rho \leq \lambda_1^1(g) \leq \lambda_1(g),$$

*where*

$$\inf \rho := \inf\{g(\rho(u), u);\ u \in T_xM,\ g(u, u) = 1,\ x \in M\}.$$

PROOF. Since id: $(M, g) \to (M, g)$ is weakly stable, and (1.27), we get

$$\begin{aligned} 0 &\leq \int_M g(J_{\mathrm{id}}V, V)\, v_g = \int_M g(\Delta_H V, V)\, v_g - 2\int_M g(\rho(V), V)\, v_g \\ &\leq \int_M g(\Delta_H V, V) - 2(\inf \rho) \int_M g(V, V)\, v_g, \end{aligned}$$

whence we get $2 \inf \rho \leq \lambda_1^1(g)$.

For the second inequality, let $f \in C^\infty(M)$ be taken as $\Delta_g f = \lambda_1(g) f$ and let $V := \operatorname{grad} f \not\equiv 0$. Then

$$\begin{aligned} \int_M g(\Delta_H V, V)\, v_g &= \int_M \langle (d\delta + \delta d) df, df \rangle\, v_g \\ &= \int_M \langle d\delta\, df, df \rangle\, v_g \\ &= \lambda_1(g) \int_M \langle df, df \rangle\, v_g \quad (\text{ since } \delta df = \Delta_g f) \\ &= \lambda_1(g) \int_M g(V, V)\, v_g. \end{aligned}$$

Thus, we get $\lambda_1^1(g) \leq \lambda_1(g)$. □

PROOF OF THEOREM (3.23). Since $(M, g)$ is a compact Kähler manifold, id: $(M, g) \to (M, g)$ is weakly stable by Corollary (3.10), so by Lemma (3.24), if $\inf \rho \geq \alpha$, then $\lambda_1(g) \geq 2\alpha$.

Conversely, assume that the equality holds. Then there exists $f \in C^\infty(M)$ such that

$$\Delta_g f = 2\alpha f$$

and $V := \operatorname{grad} f \not\equiv 0$. Then we obtain

$$\Delta_H V = 2\alpha V,$$

since for the Laplacian acting 1-forms, we have

$$\Delta_1 df = (d\delta + \delta d)df = d\delta df = 2\alpha\, df.$$

Thus, we obtain

$$2\alpha \int_M g(V, V)v_g = \int_M g(\Delta_H V, V)v_g \tag{3.25}$$
$$= \int_M g(J_{\mathrm{id}}V, V)v_g + 2\int_M g(\rho(V), V)v_g.$$

Since the eigenvalues of $J_{\mathrm{id}}$ is nonnegative, we obtain

$$\int_M g(J_{\mathrm{id}}V, V)v_g \geq 0.$$

Moreover, by assumption

$$\int_M g(\rho(V), V)v_g \geq \alpha \int_M g(V, V)v_g.$$

But together with (3.25), these two inequalities should be equalities; and thus we obtain $J_{\mathrm{id}}V = 0$. To see this, expand $V$ into the infinite sum of the eigen-vector fields as

$$V = \sum_{i=1}^{\infty} V_i, \qquad J_{\mathrm{id}}V_i = \tilde{\lambda}_i V_i, \qquad \int_M g(V_i, V_j)v_g = 0, \qquad i \neq j.$$

Let $a := \dim \mathrm{Ker}(J_{\mathrm{id}})$. Then

$$J_{\mathrm{id}}V = \sum_{i=a+1}^{\infty} \tilde{\lambda}_i V_i, \qquad \tilde{\lambda}_i > 0 \quad \forall i \geq a+1.$$

But

$$0 = \int_M g(J_{\mathrm{id}}V, V)v_g = \sum_{i=a+1}^{\infty} \tilde{\lambda}_i \int_M g(V_i, V_i)v_g$$

which yields that $V_i = 0$ $\forall i \geq a+1$. Thus, we obtain $V \in \mathrm{Ker}(J_{\mathrm{id}})$. From Corollary (3.10), we have that $\mathrm{Ker}(J_{\mathrm{id}}) \cong \mathfrak{a}(M)$, so we obtain the desired result. □

Concerning estimates of the eigenvalue $\lambda_1(g)$ of the Laplacian $\Delta_g$ acting on $C^\infty(M)$, the following theorem is well known (see also exercise 5.3).

THEOREM (3.26) (Lichnerowicz-Obata). *Let $(M, g)$ be an $m$-dimensional compact Riemannian manifold. If*

$$\inf \rho \geq \alpha = (m-1)\delta > 0,$$

*then*

$$\lambda_1(g) \geq \frac{m}{m-1}\alpha = m\delta,$$

*and equality holds if and only if $(M, g)$ is isometric to the unit sphere $(S^m, g_{S^m})$.*

REMARK. Note that $2 \geq \frac{m}{m-1}$ and $2 = \frac{m}{m-1} \Longleftrightarrow m = 2$. The estimate of Theorem (3.23) is sharper than the general one of Theorem (3.26) due to the assumption of the Käler condition. If $(M, g)$ is a compact irreducible Hermitian symmetric space, it is Einstein, i.e., $\rho = \alpha g$ and it holds that $\lambda_1(g) = 2\alpha$ (cf. [Ur4]).

## Exercises

5.1. Show the following for a compact Riemannian manifold $(M, g)$:

(i) For all $X \in \mathfrak{X}(M)$,

$$\int_M g(J_{\mathrm{id}}(X), X)\, v_g = \int_M \left\{ \frac{1}{2}|L_X g|^2 - \operatorname{div}(X)^2 \right\} v_g,$$

where $L_X$ is a (0,2)-symmetric tensor field defined by

$$(L_X g)(Y, Z) := X \cdot g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]),$$

$Y, Z \in \mathfrak{X}(M)$.

(ii) Using (i), show that if $(M, g)$ is Einstein, i.e., $\rho = c\, g$, then a sufficient and necessary condition for the identity mapping of $(M, g)$ to be weakly stable is $\lambda_1(g) \geq 2C$.

5.2. Assume that the Ricci operator $\rho$ of a compact Riemannian manifold $(M, g)$ satisfies $\rho \leq 0$, i.e., $g(\rho(X), X) \leq 0 \quad \forall X \in \mathfrak{X}(M)$. Then for all $X \in \mathfrak{X}(M)$,

$$J_{\mathrm{id}} X = 0 \Longleftrightarrow \nabla X = 0, \quad \text{i.e.,}\ \nabla_Y X = 0 \quad \forall Y \in \mathfrak{X}(M).$$

5.3. Assume that the Ricci operator of a compact Riemannian manifold $(M, g)$ satisfies

$$g(\rho(u), u) \geq \alpha\, g(u, u), \qquad u \in T_x M,\ x \in M,$$

for some positive constant $\alpha$. Show that

$$\lambda_1(g) \geq \frac{m}{m-1}\alpha,$$

where $m = \dim M$.

CHAPTER 6

# Existence, Construction, and Classification of Harmonic Maps

Problems of the existence, the explicit construction of harmonic mappings, and determining the set of all harmonic mappings between two given Riemannian manifolds have been the most important problems of differential geometry during the twenty five years since the notion of harmonic mappings was defined.

We shall explain some results concerning the problems of existence, construction and classification of harmonic mappings and show open questions about harmonic mappings in this chapter.

## §1. Existence, construction, and classification problems

One of the most fundamental problems of the theory of harmonic mappings is the following existence problem:

Let $(M, g)$, $(N, h)$ be compact Riemannian manifolds. Let $C^0(M, N)$ be the set of all continuous mappings of $M$ into $N$, and let

$$[M, N] := \{[\phi];\ \phi \in C^0(M, N)\}$$

denote the free homotopy classes of $C^0(M, N)$ (cf. (4.45) in Chapter 2). Then

(1.1) *Existence problem.* (i) For each element $\gamma$ in $[M, N]$, can one choose a harmonic mapping $\phi : (M, g) \to (N, h)$ that represents $\gamma$? That is, a given continuous mapping $\psi : M \to N$, can one deform $\psi$ continuously to a harmonic mapping $\phi : (M, g) \to (N, h)$?

(ii) In other words, we say $\gamma \in [M, N]$ is **harmonic** if one can choose a representative $\phi$ in $\gamma$ which is a harmonic mapping of $(M, g)$ into $(N, h)$. Then our question is to determine the subset of $[M, N]$ defined by

$$\{\gamma \in [M, N];\ \gamma \text{ is harmonic}\}.$$

Let us recall Theorem (4.2) in Chapter 3, which shows that if $M = S^1$, the circle, then for each Riemannian metric $h$ on $N$, each element in

$$\pi_1(N) = [S^1, N]$$

admits an energy minimizing periodic geodesic (i.e., a harmonic mapping) $\phi: S^1 \to (N, h)$. One can propose question (i) of (1.1) as a natural extension of this theorem, and if the question (i) of (1.1) is not true, our next task is (ii) of (1.1).

Eells-Wood (cf. [E.L 1]) showed if $M = T^2$ a torus and $N = S^2$ a sphere, both of dimension two, then for any Riemannian metrics on $M$ and $N$, there are no harmonic representatives for homotopy classes of degrees $\pm 1$.

One of the best answers of this problem is the following theorem.

THEOREM (1.2) (Eells-Sampson [E.S], 1964). *Let $(M, g)$, $(N, h)$ be compact Riemannian manifolds, and let $(N, h)$ have nonpositive curvature, that is, the sectional curvature ${}^N K$ of $(N, h)$ satisfies*

$$ {}^N K(u, v) \leq 0, \qquad \forall u, v \in T_y N, \ \forall y \in N. $$

*Then for any $\psi \in C^\infty(M, N)$, there exists a harmonic mapping $(M, g) \to (N, h)$ satisfying*

(i) $[\phi] = [\psi]$, i.e., *$\phi$ is homotopic to $\psi$,*

*and*

(ii) *$\phi$ is energy minimising in its homotopy class,* i.e., *for all $\psi' \in C^0(M, N)$ homotopic to $\phi$,*

$$ E(\phi) \leq E(\psi). $$

REMARK. (i) Recently, this theorem has been extended to noncompact complete Riemannian manifolds $(M, g)$, $(N, h)$ by Li-Tam [L.T].

(ii) Eells-Ferreira [E.F] showed that for compact Riemannian manifolds $(M, g)$, $(N, h)$, and any element $\gamma \in [M, N]$, there exists a Riemannian metric $\tilde{g}$ on $M$ which is conformal to $g$, i.e., $\tilde{g} = f g$ for some positive $f \in C^\infty(M)$, there exists a harmonic mapping $\phi: (M, \tilde{g}) \to (N, h)$, provided $m = \dim(M) \geq 3$. (Notice that $\phi$ and $\tilde{g}$ depend on $\gamma$.)

(1.3) *Method of Eells and Sampson* [E.S]. The method of proof in their paper in [E.S] is the so-called "(nonlinear) heat equation method", which is as follows:

To avoid the difficulty of the method of variation, they considered the following nonlinear heat equation for a given mapping $\psi \in C^\infty(M, N)$,

$$ \begin{cases} \dfrac{\partial \phi_t}{\partial t} = \tau(\phi_t), \\ \phi_0 = \psi, \end{cases} \tag{1.4} $$

for a oneparameter family of $\phi_t \in C^\infty(M, N)$ with $\phi_0 = \psi$. Here $\tau(\phi_t)$ is the tension field of $\phi_t$. They showed if $(N, h)$ has nonnegative curvature, then

(i) there exists a unique solution $\phi_t$ of (1.4) for all $0 \leq t < \infty$, and

(ii) as $t \to \infty$, the limit exists, $\lim_{t\to\infty} \phi_t = \phi_\infty$, and

(iii) the limit mapping $\phi = \phi_\infty$ is the desired harmonic mapping of $(M, g)$ into $(N, h)$.

The motivation for considering the equation (1.4) $\frac{\partial \phi_t}{\partial t} = \tau(\phi_t)$ is the following: Recall the first variation formula of the energy (cf. (1.22) in Chapter 4). For any variation $\psi_t$ of $\psi$, by defining the variation vector field $V(x) = \frac{d}{dt}\big|_{t=0}\psi_t(x) \in T_{\phi(x)}N$, $x \in M$, we get

$$\frac{d}{dt}\bigg|_{t=0} E(\psi_t) = -\int_M h(V, \tau(\phi))\, v_g. \tag{1.5}$$

Since the left-hand side of (1.5) coincides with $dE_\psi(V)$, the differentiation of the function $E$ on $C^\infty(M, N)$ at $\psi$ with respect to the direction $V$, (1.5) can be written as

$$\begin{aligned} dE_\psi(V) &= \int_M h(V, -\tau(\psi))\, v_g \\ &= (V, -\tau(\psi)). \end{aligned} \tag{1.5$'$}$$

Here we denote the global inner product ( , ) of $\Gamma(\phi^{-1}TN)$ by

$$(V, W) := \int_M h(V, W)\, v_g, \qquad V, W \in \Gamma(\phi^{-1}TN).$$

Comparing (1.5′) to (3.25) in Chapter 2 or subsection 1.6 in Chapter 3, the gradient vector field of the function $E$ on $C^\infty(M, N)$ at $\psi$, $(\nabla E)_\psi$, is just $-\tau(\psi)$:

$$(\nabla E)_\psi = -\tau(\psi), \qquad \forall \psi \in C^\infty(M, N). \tag{1.6}$$

Now we should deform $\psi$ in order to decrease the energy. For this, we may take an integral curve on $C^\infty(M, N)$ of minus the gradient vector field. We denote this integral curve by $\phi_t \in C^\infty(M, N)$, by (2.26) in Chapter 2, the equation is

$$\frac{d}{dt}\phi_t = -(\nabla E)_{\phi_t}, \qquad \phi_0 = \psi. \tag{1.7}$$

Together with (1.6), (1.7), we obtain: A deformation $\phi_t$ of $\psi$ decreasing the energy $E$ is equivalent to an integral curve $\phi_t$ of $-\nabla E$ through $\psi$, which in turn is the same as (1.4) $\frac{\partial \phi_t}{\partial t} = \tau(\phi_t)$ with $\phi_0 = \psi$. See Figure 6.1, next page.

So deforming $\psi$ in this way, then at last if $t \to \infty$, the limit mapping $\phi_\infty$ would be energy minimizing. This procedure is a key idea of their proof.

REMARK 1. To show (i), (ii), and (iii), one needs an analytical frame work. I recommend to read first [O.N], [J].

REMARK 2. The nonlinear equation (1.4) is called the **Eells-Sampson equation**. For general Riemannian manifolds $(M, g)$, $(N, h)$, the behavior of its solution has been studied recently by [Na], [C.D].

(1.8) *K. Uhlenbeck's method to prove Theorem* (1.2). This method appeared in [Uh 1], [Uh 2]:

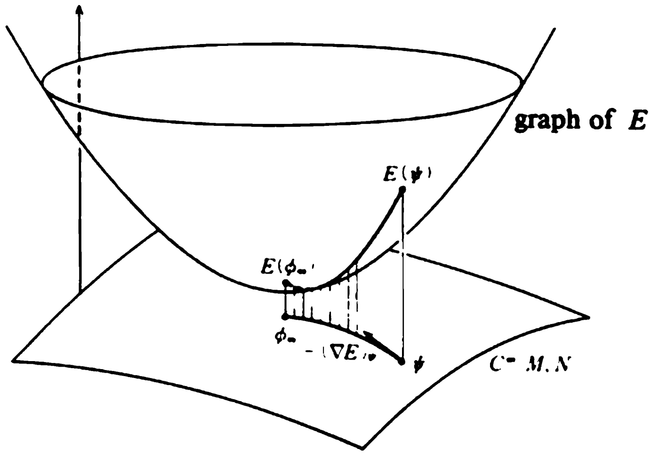



FIGURE 6.1. Graph of the function $E$ on $C^{\infty}(M, N)$.

The function $J$ on $L_{1,p}(M, N)$ in Chapter 3, satisfies the condition (C) if $1 > \frac{m}{p}$, $m = \dim M$. But $E$ does not do so. So K. Uhlenbeck considered instead $E$, the function on $L_{1,p}(M, N)$ defined by

$$E_{\epsilon} := E + \epsilon J, \qquad \epsilon > 0.$$

Then $E_{\epsilon}$ has the condition (C), inherited the good property of $J$ as far as $\epsilon > 0$. Thus, there exists a critical point of $E_{\epsilon}$, $\phi_{\epsilon} \in L_{1,p}(M, N)$ (cf. Theorem (2.17) in Chapter 3). So using the nonpositivity condition of the curvature of $(N, h)$, one can show that

(i) $\phi_{\epsilon} \in C^{\infty}(M, N)$.

(ii) As $\epsilon \to 0$, taking a subsequence $\phi_{\epsilon_i}$ of $\phi_{\epsilon}$, the limit exists, $\lim_{\epsilon_i \to 0} \phi_{\epsilon_i} = \phi_0$.

(iii) The limit mapping $\phi_0$ is the desired harmonic mapping.

This method also requires hard estimates analytically and hard arguments, but is a strong method, which weakened the assumption of the nonpositivity of the curvature of $(N, h)$ in the case $\dim M = 2$ as in Theorem (1.9). This method has been applied to prove the regularity theorem and the compactness theorem and has been applied the theory of Yang-Mills connections (cf. [Uh 3], [Uh 4]). In this book, we shall introduce a rather detailed outline of the proof of the Eells-Sampson theorem (1.2) in §4.

THEOREM (1.9) (Sacks-Uhlenbeck [S.Uh], 1981). *In the case* $\dim M = 2$, *the condition of the nonpositivity of the curvature of* $(N, h)$ *in Theorem* (1.2) *can be replaced by the condition* $\pi_2(N) = \{0\}$.

REMARK. If $(N, h)$ has nonpositive curvature, then by Hadamard's theorem, the universal covering space $\tilde{N}$ of $N$ is the Euclidean space $\mathbf{R}^n$,

$n = \dim N$, and $\pi_i(N) = \{0\}$, $i > 1$.

## §2. The case of the unit sphere

It seems that there is no general existence theory of mappings if the nonpositivity of the curvature of the target manifold $(N, h)$ is not satisfied.

The simplest case is the unit sphere with sectional curvature one, i.e., $(N, h) = (S^n, g_{S^n})$. In this case, we introduce several results about construction and classification of harmonic mappings into $(N, h)$ in the following order.

2.1. A Theorem of T. Takahashi—A method to attack the problem.

2.2. The Carmo-Wallach Theorem—Classification of harmonic mappings of any compact homogeneous Riemannian manifold with constant energy density function into the sphere.

2.3. Calabi's Theorem—Classifiction theory of all harmonic mappings of the two-dimensional sphere into the unit spheres.

2.4. Group equivariant harmonic mappings—Method of using ordinary differential equations to construct harmonic mappings with nonconstant energy density function.

**2.1. A theorem of T. Takahashi.** The only theorem to carry out the classification of harmonic mappings into the unit sphere $(S^n, g_{S^n})$ seems to be the following:

THEOREM (2.1) (T. Takahashi [T], 1966). *Let $(M, g)$ be a compact Riemannian manifold, and let $(N, h) = (S^n, g_{S^n})$ be the unit sphere with the curvature one. For a $C^\infty$-mapping $\phi : M \to S^n$, we put $\Phi := \iota \circ \phi$ and*

$$\Phi(x) = (\phi_1(x), \ldots, \phi_{n+1}(x)), \qquad x \in M,$$

*where $\iota : S^n \subset \mathbf{R}^{n+1}$ is the inclusion and $\phi_i \in C^\infty(M)$, $1 \le i \le n+1$. Then the following hold:*

(i) *A necessary and sufficient condition for $\phi : (m, g) \to (S^n, g_{S^n})$ to be harmonic is that there exists an $h \in C^\infty(M)$ with*

$$\Delta_g \phi_i = h\,\phi_i, \qquad 1 \le i \le n+1.$$

*Then this function $h \in C^\infty(M)$ is $h = 2e(\phi)$. Here $\Delta_g$ is the Laplacian of $(M, g)$ acting on $C^\infty(M)$.*

(ii) *In particular, assume that $\phi : (M, g) \to (S^n, g_{S^n})$ is an isometric immersion (cf. (3.5) in Chapter 4). Then a necessary and sufficient condition for $\phi$ to be a minimal isometric immersion is that*

$$\Delta_g \phi_i = m\,\phi_i, \qquad 1 \le i \le n+1,$$

*where $m = \dim M$.*

PROOF. If $\phi : (M, g) \to (S^n, g_{S^n})$ is an isometric immersion, then the energy density function $e(\phi) = \frac{m}{2}$, $m = \dim M$ since $\phi^* g_{S^n} = g$. Therefore, (ii) follows from (i). We already showed (i) in Corollary (2.24) in Chapter 4.

Here we give an alternative proof of (i). For this, we prepare with the following lemma:

LEMMA (2.2). *For a $C^\infty$-mapping $\phi : (M, g) \to (S^n, g_{S^n})$, we set $\Phi := \iota \circ \phi$ by the inclusion $\iota : S^n \subset \mathbf{R}^{n+1}$. Then the tension fields of $\phi$ and $\Phi$ are related to each other by*

$$\tau(\Phi)(x) = \iota_* \tau(\phi)(x) - 2e(\phi)(x)\Phi(x), \qquad x \in M,$$

*where $\Phi(x) \in S^n \subset \mathbf{R}^{n+1}$ is regarded as $\Phi(x) \in T_{\Phi(x)}S^{n\perp} \subset T_{\Phi(x)}\mathbf{R}^{n+1}$.*

PROOF. Let $\nabla$, ${}^N\nabla$, and $\nabla^0$ be the Levi-Civita connections of $(M, g)$, $(N, h) = (S^n, g_{S^n})$, $(\mathbf{R}^{n+1}, g_0)$, respectively. We denote by $A$ the second fundamental form of the inclusion $\iota : S^n \subset \mathbf{R}^{n+1}$. Then by definition we get

$$\tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i) = \sum_{i=1}^m ({}^N\nabla_{\phi_* e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i),$$

$$\tau(\Phi) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \Phi_* e_i - \Phi_* \nabla_{e_i} e_i) = \sum_{i=1}^m (\nabla^0_{\Phi_* e_i} \Phi_* e_i - \Phi_* \nabla_{e_i} e_i),$$

where $\Phi_* = \iota_* \circ \phi_*$. By the definition of $A$ we get

$$\nabla^0_{\Phi_* e_i} \Phi_* e_i = \iota_* {}^N\nabla_{\phi_* e_i} \phi_* e_i + A(\phi_* e_i, \phi_* e_i),$$

so we have

$$\begin{aligned} \tau(\Phi) &= \sum_{i=1}^m (\iota_* {}^N\nabla_{\phi_* e_i} \phi_* e_i - \iota_* \phi_* \nabla_{e_i} e_i + A(\phi_* e_i, \phi_* e_i)) \\ &= \iota_* \tau(\phi) + \sum_{i=1}^m A(\phi_* e_i, \phi_* e_i). \end{aligned}$$

By (2.16) in Chapter 4,

$$A(\phi_* e_i, \phi_* e_i) = -g_{S^n}(\phi_* e_i, \phi_* e_i)\Phi(x),$$

so we obtain

$$\begin{aligned} \sum_{i=1}^m A(\phi_* e_i, \phi_* e_i) &= -\sum_{i=1}^m g_{S^n}(\phi_* e_i, \phi_* e_i)\Phi(x) \\ &= -2e(\phi)\Phi(x) \end{aligned}$$

which yields the desired equation. □

PROOF OF THEOREM (2.1). For all $a \in \mathbf{R}^{n+1}$, define $f_a \in C^\infty(M)$ by $f_a(x) := \langle \Phi(x), a \rangle$, where $\langle \, , \, \rangle$ is the standard inner product in $\mathbf{R}^{n+1}$. For any $Y \in \mathfrak{X}(M)$, $x \in M$, letting $\sigma(t)$ be a $C^1$-curve in $M$ through $\sigma(0) = x$, $\sigma'(0) = Y_x \in T_x M$, we get

$$\begin{aligned} Y_x f_a &= \frac{d}{dt}\bigg|_{t=0} \langle \Phi(\sigma(t)), a \rangle \\ &= \left\langle \frac{d}{dt}\bigg|_{t=0} \Phi(\sigma(t)), a \right\rangle = \langle \Phi_* Y_x, a \rangle, \end{aligned}$$

where we regard $\Phi_* Y_x \in T_{\Phi(x)}\mathbf{R}^{n+1} \cong \mathbf{R}^{n+1}$. That is,

$$Y f_a = \langle \Phi_* Y, a \rangle. \tag{2.3}$$

Regarding $\Phi_* Y$ and $a$ as elements in $\Gamma(\Phi^{-1}T\mathbf{R}^{n+1})$, for $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} X(Y f_a) &= X\langle \Phi_* Y, a\rangle \\ &= \langle \tilde{\nabla}_X \Phi_* Y, a\rangle \quad (\text{ by compatibility of } h \text{ and } \tilde{\nabla}_X a = 0) \\ &= \langle \nabla^0_{\Phi_* X}\Phi_* Y, a\rangle = \langle \Phi_* \nabla_X Y + B(\Phi_* X, \Phi_* Y), a\rangle, \end{aligned} \tag{2.4}$$

where $B$ is the second fundamental form of $\Phi : (M, g) \to (\mathbf{R}^{n+1}, g_0)$ which is

$$B(\Phi_* X, \Phi_* Y) := \nabla^0_{\Phi_* X}\Phi_* Y - \Phi_* \nabla_X Y. \tag{2.5}$$

Thus, we obtain

$$\begin{aligned} \Delta_g f_a &= -\sum_{i=1}^{m} (e_i{}^2 - \nabla_{e_i} e_i) f_a \\ &= -\sum_{i=1}^{m} \langle \Phi_* \nabla_{e_i} e_i + B(\Phi_* e_i, \Phi_* e_i) - \Phi_* \nabla_{e_i} e_i, a\rangle \quad (\text{ by } (2.4), (2.5)) \\ &= -\sum_{i=1}^{m} \langle B(\Phi_* e_i, \Phi_* e_i), a\rangle \\ &= -\langle \tau(\Phi), a\rangle \qquad (\text{ by } (2.5) \text{ and definition of } \tau(\Phi)) \\ &= 2e(\phi)\langle \Phi(x), a\rangle - \langle \iota_* \tau(\phi), a\rangle \qquad (\text{ by Lemma } (2.2)) \\ &= 2e(\phi) f_a - \langle \iota_* \tau(\phi), a\rangle. \end{aligned} \tag{2.6}$$

Taking $a = e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{R}^{n+1}$,

$$\begin{aligned} \Delta_g \phi_i = h\phi_i,\ 1 \le i \le n+1 &\Longleftrightarrow \Delta_g f_a = h f_a \qquad (\forall a \in \mathbf{R}^{n+1}) \\ &\Longleftrightarrow \iota_* \tau(\phi) = (2e(\phi) - h)\Phi(x), \end{aligned}$$

from (2.6) and the definition of $f_a$. Note that

$$\iota_* \tau(\phi) \in T_{\Phi(x)} S^n, \qquad \Phi(x) \in T_{\phi(x)} S^{n\,\perp}.$$

Together with these and the injectivity of $\iota_*$, the above equation $\Delta_g \phi_i = h\,\phi_i$ holds if and only if

$$\tau(\phi) = 0$$

which then implies $h = 2e(\phi)$. □

By a theorem of T. Takahashi (2.1), the study of harmonic mappings $\phi : (M, g) \to (S^n, g_{S^n})$ is reduced to the following two cases:

$$\begin{cases} \text{(i)} & \text{the case that } e(\phi) \in C^\infty(M) \text{ is constant, that is } e(\phi) = \dfrac{\lambda}{2}, \\ \text{(ii)} & \text{all other cases.} \end{cases}$$

We treat with the case (i) in subsection 2.2. For the more difficult case (ii) we have no complete answer but we treat this in subsections 2.3, 2.4. In both cases, since $e(\phi) = \frac{1}{2}\sum_{j=1}^{n+1}\langle d\phi_j, d\phi_j\rangle$, the problem is to solve the differential system with the constraint condition in $(n+1)$-unknown functions $\phi_1, \dots, \phi_{n+1} \in C^\infty(M)$:

$$\begin{cases} \Delta_g \phi_i = \left(\sum_{j=1}^{n+1}\langle d\phi_j, d\phi_j\rangle\right)\phi_i, & 1 \le i \le n+1, \\ \sum_{i=1}^{n+1} {\phi_i}^2 = 1. \end{cases} \tag{2.7}$$

(2.8) To focus attention on harmonic mappings, we explain the relation between harmonic mappings and famous problems about minimal immersions.

(i) (**The Hsiang-Lawson problem.**) *Is the Clifford torus the only minimal embedded torus in* $(S^3, g_{S^3})$?

Here an **isometric embedding** $\phi : (M, g) \to (N, h)$ is one-to-one and an isometric immersion (cf. (3.5) in Chapter 4). A **minimal embedding** $\phi : M \to (N, h)$ is an embedding $\phi : M \to N$ such that with respect to the induced metric $g = \phi^* h$, $\phi : (M, g) \to (N, h)$ is a minimal isometric immersion (compare to (3.5) in Chapter 4). The **Clifford torus** is the minimal embedding of the flat torus $(\mathbb{R}^2/\Lambda_0, g_{\Lambda_0})$ into $(S^3, g_{S^3})$ defined by

$$\phi_0 : \mathbb{R}^2/\Lambda_0 \ni x e_1 + y e_2 \mapsto \frac{1}{\sqrt{2}}(\sin x, \cos x, \sin y, \cos y) \in S^3,$$

where $\Lambda_0 = \mathbb{Z}^2 := \{m e_1 + n e_2;\ m, n \in \mathbb{Z}\}$ and $e_1 = (1, 0)$, $e_2 = (0, 1)$ is the standard basis of $\mathbb{R}^2$. The Hsiang-Lawson problem asks whether there exist $\tau \in \mathrm{Iso}(S^3, g_{S^3})$ and $\eta \in \mathrm{Iso}(T^2, g)$ such that

$$\phi = \tau \circ \phi_0 \circ \eta$$

for any minimal isometric embedding. There are a lot of minimal isometric immersions of flat tori into the three-dimensional sphere except the Clifford torus. One of the difficulties seems to be how to make use of the embedded condition. The Hsiang-Lawson problem is reduced to:

**Problem 1.** Let $(\mathbb{R}^2/\Lambda, g_\Lambda)$ be an arbitrary flat torus. Then classify

$$\{\phi : (\mathbb{R}^2/\Lambda, g_\Lambda) \to (S^3, g_{S^3}),\ \text{harmonic mappings}\}.$$

By the uniformization theorem, for any 2-dimensional torus $(T^2, g)$, there exist a diffeomorphism $\psi : T^2 \to \mathbb{R}^2/\Lambda$ for some $\Lambda$ and a map $f \in C^\infty(T^2)$ with $f > 0$ such that $g = f\,\psi^* g_\Lambda$. Thus, identifying $(T^2, \psi^* g_\Lambda)$ and $(\mathbb{R}^2/\Lambda, g_\Lambda)$, we consider first only a Riemannian metric $g = f\, g_\Lambda$ for a positive smooth function $f \in C^\infty(\mathbb{R}^2/\Lambda)$. So let $\phi : (\mathbb{R}^2/\Lambda, g) \to (S^3, g_{S^3})$

be a minimal isometric immersion, then by a theorem of T. Takahashi (2.1) (i), we get

$$\Delta_g \phi_i = 2\phi_i, \qquad 1 \le i \le 4.$$

However, since $g = f\, g_\Lambda$,

$$\Delta_g = f^{-1} \Delta_{g_\Lambda},$$

so we obtain

$$\Delta_g \phi_i = 2f\, \phi_i, \qquad 1 \le i \le 4.$$

Again by a theorem of T. Takahashi (2.1) (i), $\phi$ is a harmonic mapping $\phi : (\mathbf{R}^2/\Lambda, g_\Lambda) \to (S^3, g_{S^3})$.

(ii) **The Hopf Problem.** Classify all tori $(\mathbf{R}^2/\Lambda, g)$ which admit isometric immersions $\Psi : (\mathbf{R}^2/\Lambda, g) \to (\mathbf{R}^3, g_0)$ with constant mean curvature $H$.

Here we explain the mean curvature. Let us recall from subsection 2.1 in Chapter 4. There we presented the second fundamental form of the closed submanifold $\iota : N \subset \mathbf{R}^3$. We can also define the second fundamental form for an isometric immersion $\Psi : (N, h) \to (\mathbf{R}^3, g_0)$ by considering $\Psi(N) \subset \mathbf{R}^3$,

$$A_y : T_yN \times T_yN \to {N_y}^{\perp} = \Psi_*(T_yN)^{\perp} \subset \mathbf{R}^3$$

for $y \in N$. Since $\dim {N_y}^{\perp} = \dim \Psi_*(T_yN)^{\perp} = 1$, $A_y$ can be regarded as a symmetric quadratic form $A_y : T_yN \times T_yN \to \mathbf{R}$. The eigenvalues $\kappa_1(y)$, $\kappa_2(y)$ of this form as a symmetric matrix are called the **principal curvatures** of the isometric immersion $\Psi : (N, h) \to (\mathbf{R}^3, g_0)$ at $y \in N$, and

$$H(y) := \frac{\kappa_1(y) + \kappa_2(y)}{2}$$

is called the **mean curvature.**

REMARK. A long standing problem has been that there is no isometric immersion $\Psi : (\mathbf{R}^2/\Lambda, g) \to (\mathbf{R}^3, g_0)$ with constant mean curvature $H$. But H. Wente [We] showed a counter example. See also Hsiang-Teng-Yu [H.T.Y 1], [H.T.Y 2], Kapouleas [Ka 1], [Ka 2], [Ka 3], Abresch [Ab], Pinkall-Sterling [P.S] and Bobenko [Bo] etc.

By a theorem of Ruh-Vilms noted below ([R.V]), the Hopf problem is reduced to the following

**Problem 2.** Classify the set of all harmonic mappings of $\phi : (\mathbf{R}^2/\Lambda) \to (S^2, g_{S^2})$.

For an isometric immersion $\Psi : (N, h) \to (\mathbf{R}^3, g_0)$, the vector $\xi(y)$ at $y \in N$ satisfying

$$\begin{cases} \xi(y) \in {N_y}^{\perp} = \Psi_*(T_yN)^{\perp} \subset \mathbf{R}^3, \\ \|\xi(y)\| = 1 \end{cases}$$

is called the **unit normal vector** of $\Psi(N)$ at $\Psi(y)$. The unit vector parallel to $\xi(y)$ of which the beginning point is the origin $(0, 0, 0) \in \mathbf{R}^3$ is denoted

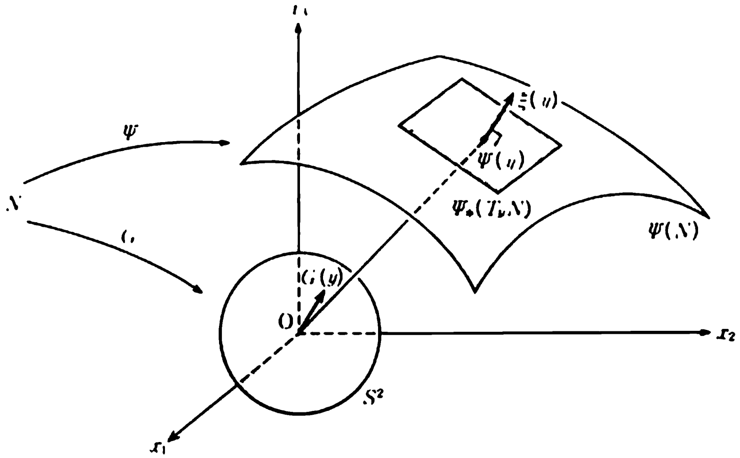



FIGURE 6.2

by $G(y)$ (see Figure 6.2). Using this unit vector, we can define a mapping, called the **Gauss mapping**,

$$G : N \ni y \mapsto G(y) \in S^2 \subset \mathbf{R}^3$$

for the immersion $\Psi : (N, h) \to (\mathbf{R}^3, g_0)$.

Then it is known that

THEOREM (Ruh-Vilms [R.V]). *For an isometric immersion* $\Psi: (N, h) \to (\mathbf{R}^3, g_0)$, *the condition that the mean curvature is constant is equivalent to the condition that the Gauss mapping* $G : (N, h) \to (S^2, g_{S^2})$ *is harmonic.*

REMARK. This theorem holds for an arbitrary isometric immersion $\Psi : (N, h) \to (\mathbf{R}^{n+1}, g_0)$ with $\dim N = n$. For the case of nonconstant mean curvature $H$, see [Kn].

Now consider $(N, h) = (\mathbf{R}^2/\Lambda, g)$ with $g = f g_\Lambda$. Then if we denote $G(y) = (G_1(y), G_2(y), G_3(y))$, $y \in N$, we have

$$\begin{aligned} G : (\mathbf{R}^2/\Lambda, f g_\Lambda) \to (S^2, g_{S^2}) \text{ is harmonic } &\Longleftrightarrow \Delta_g G_i = h\, G_i, \ 1 \le i \le 3, \\ &\Longleftrightarrow \Delta_{g_\Lambda} G_i = fh\, G_i, \ 1 \le i \le 3, \\ &\Longleftrightarrow G : (\mathbf{R}^2/\Lambda, g_\Lambda) \to (S^2, g_{S^2}) \\ &\qquad \text{is harmonic}, \end{aligned}$$

thus the Hopf problem is reduced to Problem 2.

**2.2. The do Carmo-Wallach theorem.** In this subsection, we treat the classification of harmonic mappings of $(M, g)$ into $(S^n, g_{S^n})$ with constant density function $e(\phi) = \lambda/2$ (constant). In this case, denoting $\Phi = \iota \circ \phi = (\phi_1, \dots, \phi_{n+1})$, where $\iota : S^n \subset \mathbf{R}^{n+1}$ is the inclusion, by a theorem of T.

Takahashi (2.1), each $\phi_i$, $1 \le i \le n+1$, satisfies

$$\Delta_g \phi_i = \lambda \phi_i. \tag{2.9}$$

So we may determine $(\phi_1, \dots, \phi_{n+1})$ of which $\phi_i$, $1 \le i \le n+1$ are the eigenfunctions of the Laplacian $\Delta_g$ with eigenvalue $\lambda$ and satisfy the constraint condition that

$$\sum_{i=1}^{n+1} \phi_i(x)^2 = 1, \qquad x \in M. \tag{2.10}$$

A mapping $\phi : (M, g) \to (S^n, g_{S^n})$ satisfying (2.9) and (2.10) is called the **eigen mapping**.

It is a dificult question to determine all the eigen mappings of a general compact Riemannian manifold $(M, g)$. Here we assume that $(M, g)$ is a compact homogeneous Riemannian manifold.

Then we shall prove the do Carmo-Wallach theorem which classifies completely all eigen mappings of such a manifold $(M, g)$. To state this theorem we prepare with some notation.

Let $(M, g)$ be a compact homogeneous Riemannian manifold (See subsection 4.2 in Chapter 2). Then we may assume without loss of generality that

$$M = G/K,$$

where $G$ is a compact Lie group, $K$ is its closed Lie subgroup, $G$ acts effectively on $G/K$, and the Riemannian metric $g$ is $G$-invariant (cf. (4.20) in Chapter 2).

We denote by $\mathrm{Spec}_0(M, g)$, the set of all distinct eigenvalues of $\Delta$, herafter omitting the subscript $g$ in $\Delta_g$. For $\lambda \in \mathrm{Spec}_0(M, g)$, we denote the eigenspace $V_\lambda$ corresponding to the eigenvalue $\lambda$ by

$$V_\lambda := \{f \in C^\infty(M);\ \Delta f = \lambda f\}, \qquad n(\lambda) + 1 := \dim V_\lambda.$$

Here let $C^\infty(M)$ be the set of all real-valued $C^\infty$-functions on $M$. The Lie group $G$ acts on $C^\infty(M)$ by

$$\rho(x)\, f(yK) := f \circ \tau_x{}^{-1}(yK) = f(x^{-1}yK), \qquad x, y \in G, f \in C^\infty(M) \tag{2.11}$$

which satisfies

$$\rho(x_1x_2)f = \rho(x_1)(\rho(x_2)f), \qquad \rho(x)(\lambda f_1 + \mu f_2) = \lambda\rho(x)f_1 + \mu\rho(x)f_2$$

for $x_1, x_2, x \in G$, $f, f_1, f_2 \in C^\infty(M)$, and $\lambda, \mu \in \mathbf{R}$.

(2.12) This action preserves $V_\lambda$ invariantly, i.e., if $x \in G$, $f \in V_\lambda$, then $\rho(x)f \in V_\lambda$.

Proof. In fact, since $\tau^* g = g$, $x \in G$,

$$(\Delta f) \circ \tau_x{}^{-1} = \Delta(f \circ \tau_x{}^{-1}), \qquad f \in C^\infty(M) \tag{2.13}$$

by definition of $\Delta$. Thus, if $\Delta f = \lambda f$, then

$$\Delta(\rho(x)f) = \rho(x)(\Delta f) = \lambda\, \rho(x) f. \quad \square$$

(2.14) We define the inner product $\langle\ ,\ \rangle$ on $V_\lambda$ by

$$\langle f_1, f_2\rangle := \frac{n(\lambda)+1}{\mathrm{Vol}(M, g)} \int_M f_1 f_2 v_g.$$

Then

$$\langle \rho(x) f_1, \rho(x) f_2\rangle = \langle f_1, f_2\rangle$$

for $x \in G$, $f_1, f_2 \in V_\lambda$.

PROOF. Note

$$\begin{aligned}\int_M (\rho(x)f_1)(yK)\,(\rho(x)f_2)(yK)\,v_g(yK) &= \int_M f_1(x^{-1}yK)\, f_2(x^{-1}yK)\, v_g(yK)\\ &= \int_M f_1(zK)\, f_2(zK)\, v_g(xzK)\\ &= \int_M f_1(zK)\, f_2(zK)\, v_g(zK),\end{aligned}$$

since $\tau_x{}^* v_g = v_g$ by means of $\tau_x{}^* g = g$. $\square$

Now we take an orthonormal basis for $V_\lambda$ with respect to the innner product $\langle\ ,\ \rangle$, $\{f_\lambda^i;\ i = 1, \dots, n(\lambda)+1\}$. Define the $C^\infty$-mapping $\overline{\varphi}_\lambda : M = G/K \to V_\lambda \cong \mathbb{R}^{n(\lambda)+1}$ by

$$\begin{aligned}\overline{\varphi}_\lambda(xK) : &= \sum_{i=1}^{n(\lambda)+1} f_\lambda^i(xK)\, f_\lambda^i \qquad (2.16)\\ &= {}^t(f_\lambda^1(xK), \dots, f_\lambda^{n(\lambda)+1}(xK)), \qquad x \in G,\end{aligned}$$

identifying $V_\lambda$ and $\mathbb{R}^{n(\lambda)+1}$ by means of

$$V_\lambda \ni \sum_{i=1}^{n(\lambda)+1} \xi_i f_\lambda^i \mapsto {}^t(\xi_1, \dots, \xi_{n(\lambda)+1}) \in \mathbb{R}^{n(\lambda)+1}, \qquad (2.17)$$

where ${}^t$ means the transpose. Then we obtain

LEMMA (2.18). *For any* $x \in G$, *we obtain*

$$\overline{\varphi}_\lambda(xK) \in S^{n(\lambda)+1} := \left\{ \xi = {}^t(\xi_1, \dots, \xi_{n(\lambda)+1}) \in \mathbb{R}^{n(\lambda)+1};\ \sum_{i=1}^{n(\lambda)+1} \xi_i{}^2 = 1 \right\}.$$

DEFINITION (2.19). From Lemma (2.18), we obtain the mapping $\varphi_\lambda : M \to S^{n(\lambda)}$ given by

$$\varphi_\lambda(xK) = \overline{\varphi}_\lambda(xK), \qquad x \in G.$$

By its construction, it is an eigen mapping, called the **standard eigen mapping**.

REMARK. It is not true in general that the standard eigen mapping $\varphi_\lambda : (M, g) \to (S^{n(\lambda)}, g_{S^{n(\lambda)}})$ is an isometric immersion except for the case in which $(M, g)$ the isotropy representation of $K$ on the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$ is irreducible.

PROOF OF LEMMA (2.18). Step (i). For $x \in G$, we show

$$\sum_{i=1}^{n(\lambda)+1} (f_\lambda^i(xK))^2 = \sum_{i=1}^{n(\lambda)+1} (\rho(x^{-1}) f_\lambda^i(o))^2, \qquad o = \{K\} \in G/K \tag{2.20}$$

does not depend on $x$. In fact, we can set

$$\rho(x^{-1}) f_\lambda^i = \sum_{j=1}^{n(\lambda)+1} a_{ij}(x) f_\lambda^j,$$

where $A = (a_{ij}(x)) \in O(n(\lambda)+1)$ for all $x \in G$. And then the right-hand side of (2.20) coincides with

$$\sum_{j,k=1}^{n(\lambda)+1} \left( \sum_{i=1}^{n(\lambda)+1} a_{ij}(x) a_{ik}(x) \right) f_\lambda^j(o) f_\lambda^k(o) = \sum_{j=1}^{n(\lambda)+1} (f_\lambda^j(o))^2,$$

which is a constant, say $C$.

Step (ii). Integrate (2.20) in $xK$ over $M$, and multiply by $\frac{n(\lambda)+1}{\mathrm{Vol}(M,g)}$, then we get

$$C(n(\lambda)+1) = \sum_{i=1}^{n(\lambda)+1} (f_\lambda^i, f_\lambda^i) = n(\lambda)+1;$$

thus, we get $C = 1$. □

Let $o = {}^t(1, 0, \dots, 0) \in S^{n(\lambda)}$ be the origin of the unit sphere, and choose an orthogonal matrix $A$ of degree $n(\lambda)+1$ in such a way that

$$A\varphi_\lambda(eK) = o = {}^t(1, 0, \dots, 0), \tag{2.21}$$

using $\varphi_\lambda(eK) \in S^{n(\lambda)}$ and Example 3 (4.23) in Chapter 2. Define an alternative orthonormal basis for $V_\lambda$ by

$$f_\lambda'^i := \sum_{j=1}^{n(\lambda)+1} A_{ij} f_\lambda^j,$$

and define the corresponding mapping

$$\begin{aligned} \overline{\varphi}_\lambda'(xK) : &= \sum_{i=1}^{n(\lambda)+1} f_\lambda'^i(xK) f_\lambda'^i \\ &= {}^t(f_\lambda'^1(xK), \dots, f_\lambda'^{n(\lambda)+1}(xK)) \end{aligned}$$

which induces the eigen mapping $\varphi_\lambda'$ of $(M, g)$ into $S^{n(\lambda)}$. Then we obtain

LEMMA (2.22). *The eigen mapping $\varphi'_\lambda$ of $(M, g)$ into $(S^{n(\lambda)}, g_{S^{n(\lambda)}})$ satisfies*

(i) $\varphi'_\lambda = A \circ \varphi_\lambda$,

(ii) $\varphi'_\lambda(x^{-1}K) = \rho(x)v_0$, $x \in G$,

*where* $(A \circ \varphi_\lambda)(xK) := A\varphi_\lambda(xK)$, $x \in G$, *and* $v_0 := f'^1_\lambda \in V_\lambda$.

PROOF. For (i), we calculate

$$\begin{aligned}\vec{\varphi}'_\lambda(xK) &= {}^t(f'^1_\lambda(xK), \dots, f'^{n(\lambda)+1}_\lambda(xK)) \\ &= A\,{}^t(f^1_\lambda(xK), \dots, f^{n(\lambda)+1}_\lambda(xK)),\end{aligned}$$

which denotes the multiplication of the matrix $A = (A_{ij})$ and the $n$th column vector; that is, $f'^i_\lambda = \sum_{j=1}^{n(\lambda)+1} A_{ij} f^j_\lambda$. Thus, we get (i). For (ii),

$$\begin{aligned}\vec{\varphi}'_\lambda(x^{-1}K) &= \rho(x)\vec{\varphi}'_\lambda(eK) \\ &= \rho(x)\sum_{i=1}^{n(\lambda)+1} f'^i_\lambda(eK)\, f'^i_\lambda \\ &= \rho(x)\sum_{i,j=1}^{n(\lambda)+1} A_{ij} f^j_\lambda(eK)\, f'^i_\lambda.\end{aligned} \tag{2.23}$$

Here note that equation (2.21) is equivalent to

$$A\,{}^t(f^1_\lambda(eK), \dots, f^{n(\lambda)+1}_\lambda(eK)) = {}^t(1, 0, \dots, 0)$$

and to

$$\sum_{j=1}^{n(\lambda)+1} A_{ij} f^j_\lambda(eK) = \begin{cases} 1, & (i = 1), \\ 0 & (i \neq 1). \end{cases}$$

Thus, from (2.23), we get

$$\vec{\varphi}'_\lambda(x^{-1}K) = \rho(x) f'^1_\lambda = \rho(x)v_0. \quad \square$$

DEFINITION (2.24). (i) Two harmonic mappings $\phi$, $\psi$: $(M, g) \to (S^n, g_{S^n})$ are **image equivalent** if there exists an element $\tau \in \mathrm{SO}(n+1)$ such that

$$\psi = \tau \circ \phi,$$

in which case we write $\varphi \sim \psi$.

(ii) A harmonic mapping $\phi : (M, g) \to (S^n, g_{S^n})$ is said to be **full** if the image $\phi(M)$ is contained in no $(n-1)$-dimensional hypersphere of $S^n$.

THEOREM (2.25) (do Carmo-Wallach [D.W]). *Let $(M, g)$ be an arbitrary compact homogeneous Riemannian manifold. Then*

(i) *assume that $\phi : (M, g) \to (S^n, g_{S^n})$ is a full eigen mapping with $e(\phi) = \frac{\lambda}{2}$. Then $\lambda \in \mathrm{Spec}_0(M, g)$ and $n \le n(\lambda)$.*

(ii) *Let $\mathscr{A}_\lambda(M)$ be the set of all equivalence classes $[\phi]$ of full eigen mappings of $(M, g)$ into $(S^n, g_{S^n})$. Then $\mathscr{A}_\lambda(M)$ corresponds one-to-one onto a* *compact convex body $L_\lambda$ of some Euclidean space $E_\lambda$, and each interior point of $L_\lambda$ corresponds to a full eigen mapping of $(M, g)$ onto $(S^{n(\lambda)}, g_{S^{n(\lambda)}})$, and the boundary one of $L_\lambda$ corresponds to a full eigen mapping of $(M, g)$ into $(S^n, g_{S^n})$ for some $n < n(\lambda)$.*

(2.26) We explain the correspondance between $L_\lambda$ and the eigen mappings in Theorem (2.25) (ii) and $E_\lambda$:

(i) Let $S^2(V_\lambda)$ be the symmetric product of the vector space $V_\lambda$. That is,

$$S^2(V_\lambda) := \{u \cdot v;\ u,\ v \in V_\lambda\} \subset V_\lambda \otimes V_\lambda,$$

where $u \cdot v := \frac{1}{2}(u \otimes v + v \otimes u)$, $u$, $v \in V_\lambda$. Then $G$ acts on $S^2(V_\lambda)$ by

$$\rho(x)(u \cdot v) := \rho(x)u \cdot \rho(x)v.$$

The inner product $\langle\ ,\ \rangle$ on $V_\lambda$ is canonically extended to $S^2(V_\lambda)$ as follows. Taking an orthonormal basis $\{v_i\}_{i=1}^{n(\lambda)+1}$, we give an inner product $\langle\ ,\ \rangle$ on $V_\lambda \otimes V_\lambda$ in such a way that $\{v_i \otimes v_j;\ 1 \le i, j \le n(\lambda)+1\}$ gives an orthonormal basis of $V_\lambda \otimes V_\lambda$, and then we give the same inner product restricted to the subspace $S^2(V_\lambda)$ of $V_\lambda \otimes V_\lambda$, denoted by the same symbol $\langle\ ,\ \rangle$. Then

$$\langle \rho(x)w,\ \rho(x)w' \rangle = \langle w,\ w' \rangle, \qquad w,\ w' \in S^2(V_\lambda),\ x \in G.$$

(ii) Let $L(V_\lambda, V_\lambda)$ be the set of all linear mappings of $V_\lambda$ into itself, and put

$$\mathscr{S} := \{A \in L(V_\lambda, V_\lambda);\ \langle Au, v \rangle = \langle u, Av \rangle,\ u,\ v \in V_\lambda\}.$$

We can identify $\mathscr{S}$ and $S^2(V_\lambda)$ as

$$u \cdot v := \frac{1}{2}\{\langle u, t \rangle v + \langle v, t \rangle u\}, \qquad u,\ v,\ t \in V_\lambda.$$

Under this identification, the inner product on $\mathscr{S}$, denoted by $\ll\ ,\ \gg$ corresponding to the inner product $\langle\ ,\ \rangle$ on $S^2(V_\lambda)$ is

$$\ll A, B \gg = \operatorname{tr}(AB), \qquad A,\ B \in \mathscr{S},$$

and the $G$-action on $S(V_\lambda)$ induces the one on $\mathscr{S}$ by

$$\rho(x)A := \rho(x) \circ A \circ \rho(x)^{-1}, \qquad x \in G,\ A \in \mathscr{S}.$$

Proof. In fact, for the inner product, note that

$$\{\sqrt{2}v_i \cdot v_j;\ 1 \le i < j \le n(\lambda) + 1,\ v_i \cdot v_i;\ 1 \le i \le n(\lambda) + 1\}$$

is an orthonormal basis for $\langle S^2(V_\lambda), \langle\ ,\ \rangle\rangle$. Moreover, if we express each element of $S^2(V_\lambda)$ by a matrix using the basis $\{v_i\}_{i=1}^{n(\lambda)+1}$, the above identification is given as

$$\sqrt{2}v_i \cdot v_j \leftrightarrow \frac{1}{\sqrt{2}} \begin{matrix} \\ i \\ j \end{matrix}\begin{pmatrix} & i & & j & \\ \hline & & & 1 & \\ & 1 & & & \end{pmatrix}, \qquad v_i \cdot v_i \leftrightarrow \begin{matrix} \\ i \end{matrix}\begin{pmatrix} & i & \\ & 1 & \end{pmatrix}.$$

Here each entry of the matrices is zero except 1. Under the identification of this corespondance, we can show $\ll A, B \gg = \mathrm{tr}(AB)$.

Since

$$\begin{aligned} \rho(x)u \cdot \rho(x)v\,(t) &= \frac{1}{2}(\langle \rho(x)u, t\rangle\, \rho(x)v + \langle \rho(x)v, t\rangle\, \rho(x)u) \\ &= \frac{1}{2}(\langle u, \rho(x^{-1})t\rangle\, \rho(x)v + \langle v, \rho(x^{-1})t\rangle\, \rho(x)u) \\ &= \rho(x) \circ (u \cdot v) \circ \rho(x^{-1})\,(t), \end{aligned}$$

we obtain

$$\rho(x)A = \rho(x) \circ A \circ \rho(x^{-1}), \qquad A \in \mathscr{S}. \ \square$$

Furthermore, we obtain:

$$\langle Au, u\rangle = \ll A, u \cdot u \gg, \qquad u \in V_\lambda,\ A \in \mathscr{S}. \tag{2.27}$$

PROOF. Since $u \cdot u\,(t) = \langle u, t\rangle\, u$, we get

$$\langle Au, u\rangle = \sum_{i=1}^{n(\lambda)+1} \langle Au, v_i\rangle \langle u, v_i\rangle,$$

$$\begin{aligned} \ll A, u \cdot u \gg &= \mathrm{tr}(A \circ (u \cdot u)) = \sum_{i=1}^{n(\lambda)+1} \langle A((u \cdot u)(v_i)), v_i\rangle \\ &= \sum_{i=1}^{n(\lambda)+1} \langle Au, v_i\rangle\langle u, v_i\rangle \end{aligned}$$

which implies (2.27). $\square$

With this preparation, the $E_\lambda$, $L_\lambda$ are given as follows: Let $W_0$ be a subspace of $S^2(V_\lambda)$ generated by elements $\rho(x)(v_0 \cdot v_0)$, $x \in G$, and denote by the same letter $W_0$ the corresponding subspace in $\mathscr{S}$ under the above identification. Then we define

$$E_\lambda := \{A \in \mathscr{S};\ \ll A, B \gg = 0,\ \forall B \in W_0\}, \tag{2.28}$$

$$L_\lambda := \{C \in E_\lambda;\ C + I \geq 0\}, \tag{2.29}$$

where $I$ is the identity mapping of $V_\lambda$ and for $B \in \mathscr{S}$, $B \geq 0$ means semipositive definite, that is,

$$\langle Bu, u\rangle \geq 0, \qquad \forall u \in V_\lambda.$$

The identification in Theorem (2.25) is given by

$$L_\lambda \ni C \mapsto (C + I)^{1/2} \circ \varphi'_\lambda. \tag{2.30}$$

(2.31) PROOF OF THEOREM (2.25). (i) Let $\phi : (M, g) \to (S^n, g_{S^n})$ be a full eigen mapping with $e(\phi) = \frac{\lambda}{2}$. Then by a theorem of T. Takahashi (2.21), denoting $\iota \circ \phi = (\phi_1, \ldots, \phi_{n+1})$, we get

$$\Delta\phi_i = \lambda\phi_i, \qquad 1 \leq i \leq n + 1.$$

That is, $\phi_i \in V_\lambda$, $1 \le i \le n+1$. Since $\phi$ is full, $\{\phi_1, \dots, \phi_{n+1}\}$ is linear independent. Thus, we get

$$n+1 \le \dim V_\lambda = n(\lambda)+1, \quad \text{i.e.,}\ n \le n(\lambda).$$

(ii) By (i), each $\phi_i$ can be expressed by a linear combination of $\{f_\lambda'^j\,;\ 1 \le j \le n(\lambda)+1\}$. Therefore, adding $(n(\lambda)-n)$ zeros if necessary, we can choose a matrix $A$ of degree $n(\lambda)+1$ in such a way that

$${}^t(\phi_1, \dots, \phi_{n+1}, 0, \dots, 0) = A\,{}^t(f_\lambda'^1, \dots, f_\lambda'^{n(\lambda)+1}).$$

Here we take the polar decomposition of $A$, i.e.,

$$A = TB,$$

where $T$ is an orthogonal matrix, $B$ is a semipositive definite symmetric matrix. Then denoting the inclusion by $j : S^n \subset S^{n(\lambda)}$, we get the following image equivalence relation

$$j \circ \phi \sim A\phi'_\lambda \sim B\phi'_\lambda.$$

Here we give the condition for the $C^\infty$-mapping $B\phi'_\lambda : M \to \mathbf{R}^{n(\lambda)+1}$ to be a mapping into the sphere $S^{n(\lambda)}$. Since

$$\phi'_\lambda(x^{-1}K) = \rho(x)v_0, \qquad x \in G,$$

by Lemma (2.22), we get for all $x \in G$,

$$\begin{aligned} 1 &= \langle B\phi'_\lambda(xK), B\phi'_\lambda(xK)\rangle = \langle B\rho(x)v_0, B\rho(x)v_0\rangle \qquad (2.32)\\ &= \langle B^2\rho(x)v_0, \rho(x)v_0\rangle \qquad (B \text{ is symmetric })\\ &= \ll B^2, \rho(x)v_0 \cdot \rho(x)v_0 \gg \qquad (\text{by } (2.27))\\ &= \ll B^2, \rho(x)(v_0 \cdot v_0) \gg . \end{aligned}$$

On the other hand, by (2.27) and the $G$-invariance of $\ll\ ,\ \gg$, for all $x \in G$, we get

$$\ll I, \rho(x)(v_0\,, \cdot v_0) \gg = \ll I, v_0 \cdot v_0 \gg = \langle v_0, v_0\rangle = 1. \qquad (2.33)$$

Thus, by (2.32) and (2.33), a necessary and sufficient condition for $B\phi'_\lambda(M) \subset S^{n(\lambda)}$ is

$$\ll B^2 - I, \rho(x)(v_0 \cdot v_0) \gg = 0, \qquad \forall x \in G \qquad (2.34)$$

which is equivalent to

$$C := B^2 - I \in E_\lambda.$$

Conversely, $C \in E_\lambda$ satisfies $C + I \ge 0$, then putting $B := (C+I)^{1/2}$, we have that $\phi := B \circ \phi'_\lambda : (M, g) \to (S^n, g_{S^n})$, $n \le n(\lambda)$ is a full eigen mapping with $e(\phi) = \frac{\lambda}{2}$. We obtain Theorem (2.25). □

REMARK 1. Consider

$$\mathscr{B}(M) := \{[\phi];\ \phi : (M, g) \to (S^n, g_{S^n}),\ \text{minimal isometric immersions}\}.$$

Since $e(\phi) = \frac{m}{2}$, $m = \dim M$, we obtain

$$\mathscr{B}(M) = \{[\phi];\ \phi \in \mathscr{A}_m(M) \text{ and } \phi^* g_{S^n} = g\} \subset \mathscr{A}_m(M).$$

Thus, we can, in priciple, determine $\mathscr{B}(M)$ by Theorem (2.25), but its calculation is rather complicated (cf. [P.Ur], [Ur3]).

REMARK 2. From the definition of $L_\lambda$, it follows that $L_\lambda$ is a compact convex body in $E_\lambda$, i.e., if $C_1, C_2 \in L_\lambda$, then $tC_1 + (1-t)C_2 \in L_\lambda$ for all $0 \le t \le 1$.

To show that $L_\lambda$ is compact, it suffices to show that all the eigenvalues of $C \in L_\lambda$ are bounded. To see this, note that

$$\begin{aligned}\int_M \overrightarrow{\varphi}_\lambda(xK) \cdot \overrightarrow{\varphi}_\lambda(xK)\, v_g &= \sum_{i,j=1}^{n(\lambda)+1} \left(\int_M f_\lambda^i(xK) f_\lambda^j(xK)\, v_g\right) f_\lambda^i \cdot f_\lambda^j \\ &= \frac{\mathrm{Vol}(M, g)}{n(\lambda)+1} \sum_{i=1}^{n(\lambda)+1} f_\lambda^i \cdot f_\lambda^i,\end{aligned}$$

since $\overrightarrow{\varphi}_\lambda(xK) = \sum_{i=1}^{n(\lambda)+1} f_\lambda^i(xK)\, f_\lambda^i$.

For $C \in L_\lambda$, putting $C = B^2 - I$, we obtain by (2.34),

$$\ll C, \overrightarrow{\varphi}_\lambda(xK) \cdot \overrightarrow{\varphi}_\lambda(xK) \gg = 0, \qquad \forall x \in G,$$

since $\rho(x)(v_0 \cdot v_0) = \overrightarrow{\varphi}_\lambda(xK) \cdot \overrightarrow{\varphi}_\lambda(xK)$ (cf. Lemma (2.22) (ii)). Integrate the above over $M$ in $x$ to obtain

$$\mathrm{tr}(C) = \sum_{i=1}^{n(\lambda)+1} (C f_\lambda^i, f_\lambda^i) = \ll C, \sum_{i=1}^{n(\lambda)+1} f_\lambda^i \cdot f_\lambda^i \gg = 0,$$

from (2.27). Thus, since $C + I \ge 0$, denoting the eigenvalues of $C$ by $\lambda_1, \dots, \lambda_{n(\lambda)+1}$, we obtain

$$\sum_{i=1}^{n(\lambda)+1} \lambda_i = 0 \quad \text{and} \quad \lambda_i + 1 \ge 0, \qquad i = 1, \dots, n(\lambda)+1.$$

Therefore, we obtain

$$-1 \le \lambda_i \le n(\lambda), \qquad 1 \le i \le n(\lambda)+1$$

which yields the compactness of $L_\lambda$. □

In particular, let $(M, g) = (\mathbf{R}^m/\Lambda, g_\Lambda)$ be a flat torus. Then for the Laplacian $\Delta$,

$$\begin{cases} \text{the eigenvalues are } 4\pi^2 |\mathbf{n}|^2, \ \mathbf{n} \in \Lambda^*, \\ \text{the eigenfunctions are } \mathbf{R}^m/\Lambda \ni \pi(x) \mapsto e^{2\pi i (\mathbf{n}, x)}, \end{cases}$$

where $\Lambda^* := \{\mathbf{n} \in \mathbf{R}^m;\ (\mathbf{n}, x) \in \mathbf{Z},\ \forall x \in \Lambda\}$. Thus, the multiplicity of any eigenvalue $\lambda$ is always even, say $2p$, and then we may write

$$\{\mathbf{n} \in \Lambda^*;\ 4\pi^2 \|\mathbf{n}\|^2 = \lambda\} = \{\mathbf{n}_1, \dots, \mathbf{n}_p, -\mathbf{n}_1, \dots, -\mathbf{n}_p\}.$$

Then by Theorem (2.25), we obtain

COROLLARY (2.35). *Let* $(M, g) = (\mathbb{R}^m/\Lambda, g_\Lambda)$ *be a flat torus. Then*

(i) $\dim \mathscr{A}_\lambda(M) := \dim E_\lambda = 2p^2+p-1-2N \geq p-1$, *where* $N$ *is the number of all distinct elements of* $\{\mathbf{n}_j+\mathbf{n}_k\ (1 \leq j \leq k \leq p),\ \mathbf{n}_j-\mathbf{n}_k\ (1 \leq j < k \leq p)\}$.

(ii) $\mathscr{A}_\lambda(M) \supset \mathscr{A}_{\lambda,0}(M)$, *where* $\mathscr{A}_{\lambda,0}(M)$ *is the set of all image equivalence classes of full eigen mappings of the following*:

$$\mathbb{R}^m/\Lambda \ni \pi(x) \mapsto \frac{1}{\sqrt{p}}\left(\sqrt{a_1+1}\, e^{2\pi i \langle \mathbf{n}_1, x\rangle}, \dots, \sqrt{a_p+1}\, e^{2\pi i \langle \mathbf{n}_p, x\rangle}\right),$$

*where the* $a_1, \dots, a_p$ *run over all real numbers satisfying*

$$a_i + 1 \geq 0\ (1 \leq i \leq p) \quad and \quad \sum_{i=1}^{p} a_i = 0.$$

(iii) *In particular, if* $\dim M = \dim \mathbb{R}^m/\Lambda = m = 2$, *then* $\mathscr{A}_\lambda(M) = \mathscr{A}_{\lambda,0}(M)$.

(iv) (*Hsiang-Lawson, Sasaki* [Ss]) *The only minimal embedded* **flat** *torus is the Clifford torus.*

The proof of Corollary (2.35) is omitted. See [P.Ur].

**2.3. Calabi's Theorem.** E. Calabi showed that the set of all full harmonic mappings of $(S^2, g_{S^2})$ into $(S^n, g_{S^n})$ corresponds to the set of all full holomorphic mappings of $P^1(\mathbb{C})$ into some compact Kähler manifold, which is a usefull theorem because holomorphic mappings are much easier to study. In this subsection, we introduce his theory. The following proposition is crucial.

PROPOSITION (2.36). *Let* $(M, g)$, $(N, h)$, $(Y, g')$ *be three compact Riemannian manifolds. Let* $\pi : (Y, g') \to (N, h)$ *be the Riemannian submersion* (*cf.* (3.7) *in Chapter* 4). *That is,* $\pi : Y \to N$ *is an onto mapping, for each* $y \in Y$, *the differentiation* $\pi_* : T_yY \to T_{\pi(y)}N$ *is surjective,* $V_y = \mathrm{Ker}(\pi_*)$, *and* $\pi_* : (H_y, g'_y) \to (T_{\pi(y)}N, h_{\pi(y)})$ *is isometric, where* $T_yY = V_y \oplus H_y$ *is the orthogonal decomposition with respect to* $g'_y$.

*For* $\psi \in C^\infty(M, Y)$, *put* $\phi := \pi \circ \psi \in C^\infty(M, N)$. *Assume that*

(i) $\psi : (M, g) \to (Y, g')$ *is a harmonic mapping,*

*and*

(ii) $\psi$ *is* **horizontal**, *i.e.*,

$$\psi_{*x}(T_xM) \subset H_{\psi(x)}, \qquad x \in M.$$

*Then* $\phi = \pi \circ \psi : (M, g) \to (N, h)$ *is a harmonic mapping.*

$$\begin{array}{ccc} & & (Y, g') \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ (M, g) & \overset{\phi}{\longrightarrow} & (N, h) \end{array}$$

PROOF. Let $\nabla$, ${}^N\nabla$, ${}^Y\nabla$ be the Levi-Civita connections of $(M, g)$, $(N, h)$, $(Y, g')$, respectively. Since $\phi = \pi \circ \psi$, $\phi_* = \pi_* \circ \psi_*$. Taking an orthonormal local frame field $\{e_i\}_{i=1}^m$ on $(M, g)$, $\psi_* e_i$ is horizontal by the assumption, due to O'Neill's formula (cf. (3.9) in Chapter 4), we get

$$^N\nabla_{\phi_* e_i}\phi_* e_i = {}^N\nabla_{\pi_*\psi_* e_i}\pi_*\psi_* e_i = \pi_* {}^Y\nabla_{\psi_* e_i}\psi_* e_i.$$

Thus, we obtain

$$\begin{aligned}\tau(\phi) &= \pi_* \sum_{i=1}^m ({}^Y\nabla_{\psi_* e_i}\psi_* e_i - \psi_* \nabla_{e_i} e_i) \\ &= \pi_* \tau(\psi) = 0. \quad \square\end{aligned}$$

To state Calabi's theorem, we prepare with the following:

Let $(\mathbf{R}^{2p+1}, (\ ,\ ))$ be the $(2p+1)$-dimensional Euclidean space. We extend the inner product $(\ ,\ )$ to a complex bilinear form $(\ ,\ )_{\mathbf{C}}$ on the complex Euclidean space $\mathbf{C}^{2p+1}$ by

$$\begin{aligned}(x + \sqrt{-1}y, u + \sqrt{-1}v)_{\mathbf{C}} := (x, u) - (y, v) \\ + \sqrt{-1}((y, u) + (x, v)),\end{aligned}$$

for $x, y, u, v \in \mathbf{R}^{2p+1}$. A complex subspace $V$ of $\mathbf{C}^{2p+1}$ is called **isotropic** if it satisfies

$$(v, w)_{\mathbf{C}} = 0, \qquad \forall v, w \in V,$$

and we denote by $\mathscr{I}_p$, the set of all $p$-dimensional isotropic complex subspaces in $\mathbf{C}^{2p+1}$. Then

LEMMA (2.37). (i) *The special orthogonal groups and the unitary group have the following inclusion relations:*

$$\mathrm{U}(p) \subset \mathrm{SO}(2p) \subset \mathrm{SO}(2p+1),$$

*where*

$$\mathrm{U}(p) \ni a + \sqrt{-1}b \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SO}(2p),$$

$$\mathrm{SO}(2p) \ni x \mapsto \begin{pmatrix} x & 0 \\ {}^t0 & 1 \end{pmatrix} \in \mathrm{SO}(2p+1).$$

(ii) $\mathrm{SO}(2p+1)$ *acts transitively on* $\mathscr{I}_p$, *and we get*

$$\mathscr{I}_p = \mathrm{SO}(2p+1)/\mathrm{U}(p).$$

(iii) *The inclusion in* (i) *induces the canonical mapping*

$$\pi : \mathrm{SO}(2p+1)/\mathrm{U}(p) \ni x\mathrm{U}(p) \mapsto x\mathrm{SO}(2p) \in \mathrm{SO}(2p+1)/\mathrm{SO}(2p)$$

*and the Riemannian submersion* $\pi : (\mathscr{I}_p, g') \to (S^{2p}, g_{S^{2p}})$. *Here the Riemannian metric* $g'$ *on* $\mathscr{I}_p$ *is the* $\mathrm{SO}(2p+1)$*-invariant one.*

(iv) *$\mathcal{J}_p$ admits a canonical complex structure in such a way that each translation $\tau_x$, $x \in SO(2p+1)$ is holomorphic, and $(\mathcal{J}_p, g')$ is a Kähler manifold.*

REMARK. If $p = 2$, $\mathcal{J}_2 = SO(5)/U(2) = P^3(\mathbb{C})$, and we get the Riemannian submersion

$$\pi : (P^3(\mathbb{C}), g') \to (S^4, g_{S^4})$$

which is called the **Calabi-Penrose twister fiber bundle (fibering)**.

PROOF OF LEMMA (2.37). (i) is clear. (ii) The standard Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^{2p+1}$ is

$$\langle v, w\rangle = \langle v, \overline{w}\rangle_{\mathbb{C}}, \qquad v, w \in \mathbb{C}^{2p+1}.$$

Here $\overline{w}$ is the complex conjugate of $w \in \mathbb{C}^{2p+1}$. We take any $V \in \mathcal{J}_p$ and let $\{z_1, \dots, z_p\}$ be an orthonormal basis of $V$ with respect to $\langle\ ,\ \rangle$. For $k = 1, \dots, p$, put

$$z_k = \frac{1}{\sqrt{2}}(X_k + \sqrt{-1}Y_k), \qquad X_k, Y_k \in \mathbb{R}^{2p+1}.$$

We get that $\{X_k, Y_k;\ 1 \le k \le p\}$ is orthonormal with respect to $\langle\ ,\ \rangle$. Because

$$\begin{aligned}\delta_{k\ell} = \langle z_k, z_\ell\rangle = \langle z_k, \overline{z}_\ell\rangle_{\mathbb{C}} &= \frac{1}{2}\langle X_k + \sqrt{-1}Y_k, X_\ell - \sqrt{-1}Y_\ell\rangle_{\mathbb{C}} \\ &= \frac{1}{2}\{\langle X_k, X_\ell\rangle + \langle Y_k, Y_\ell\rangle + \sqrt{-1}(\langle Y_k, X_\ell\rangle - \langle X_k, Y_\ell\rangle)\}.\end{aligned}$$

Since $V$ is isotropic, we get

$$\begin{aligned}0 = \langle z_k, z_\ell\rangle_{\mathbb{C}} &= \frac{1}{2}\langle X_k + \sqrt{-1}Y_k, X_\ell + \sqrt{-1}Y_\ell\rangle_{\mathbb{C}} \\ &= \frac{1}{2}\{\langle X_k, X_\ell\rangle - \langle Y_k, Y_\ell\rangle + \sqrt{-1}(\langle Y_k, X_\ell\rangle + \langle X_k, Y_\ell\rangle)\}.\end{aligned}$$

Together with these equalities,

$$\begin{cases}\langle X_k, X_\ell\rangle + \langle Y_k, Y_\ell\rangle = 2\delta_{k\ell}, & \langle Y_k, X_\ell\rangle - \langle X_k, Y_\ell\rangle = 0, \\ \langle X_k, X_\ell\rangle - \langle Y_k, Y_\ell\rangle = 0, & \langle Y_k, X_\ell\rangle + \langle X_k, Y_\ell\rangle = 0,\end{cases}$$

which yields

$$\langle X_k, X_\ell\rangle = \langle Y_k, Y_\ell\rangle = \delta_{k\ell}, \qquad \langle X_k, Y_\ell\rangle = 0. \tag{2.38}$$

Take an element $Z \in \mathbb{R}^{2p+1}$ such that $\{X_1, \dots, X_p, Y_1, \dots, Y_p, Z\}$ is an orthonormal basis for $\mathbb{R}^{2p+1}$ with respect to $\langle\ ,\ \rangle$. Let $\{e_1, \dots, e_{2p+1}\}$ be the standard basis for $\mathbb{R}^{2p+1}$, and set

$$V_0 := \left\{\sum_{k=1}^{p} a_k(e_k + \sqrt{-1}e_{p+k});\ a_k \in \mathbb{C},\ 1 \le k \le p\right\}.$$

Then $V_0 \in \mathcal{J}_p$. Because $z_k = \frac{1}{\sqrt{2}}(e_k + \sqrt{-1}e_{p+k})$ satisfies (2.38), which implies $0 = (z_k, z_\ell)_{\mathbf{C}}$.

For the two orthonormal bases $\{X_1, \dots, X_p, Y_1, \dots, Y_p, Z\}$ and $\{e_1, \dots, e_{2p+1}\}$, we can choose an orthogonal matrix $T$ such that

$$\{X_1, \dots, X_p, Y_1, \dots, Y_p, Z\} = T\{e_1, \dots, e_{2p+1}\}.$$

But we may assume $T \in \mathrm{SO}(2p+1)$ if we consider an alternative basis $\{X_1, \dots, X_p, Y_1, \dots, Y_p, -Z\}$ when $\det T = -1$. Then for $k = 1, \dots, p$,

$$\begin{aligned} z_k &= X_k + \sqrt{-1}Y_k \\ &= Te_k + \sqrt{-1}Te_{p+k} = T(e_k + \sqrt{-1}e_{p+k}), \end{aligned}$$

so that we obtain $V = TV_0$, $T \in \mathrm{SO}(2p+1)$. Thus $\mathrm{SO}(2p+1)$ acts transitively on $\mathcal{J}_p$. To see that the isotropy subgroup of $\mathrm{SO}(2p+1)$ at $V_0$ is $\mathrm{U}(p)$, we note that for any $T \in \mathrm{SO}(2p+1)$,

$$TV_0 = V_0 \Longleftrightarrow \begin{cases} T(e_\ell + \sqrt{-1}e_{p+\ell}) = \sum_{k=1}^{p} c_{k\ell}(e_k + \sqrt{-1}e_{p+k}), & c_{k\ell} \in \mathbf{C}, \\ Te_{2p+1} = e_{2p+1}. \end{cases}$$

Here $c_{k\ell}$ denotes $a_{k\ell} + \sqrt{-1}b_{k\ell}$ with $a_{k\ell}, b_{k\ell} \in \mathbf{R}$. Then we get

$$\begin{aligned} TV_0 = V_0 &\Longleftrightarrow \begin{cases} Te_{2p+1} = e_{2p+1}, \quad Te_\ell = \sum_{k=1}^{p}\{a_{k\ell}e_k - b_{k\ell}e_{p+k}\}, \\ Te_{p+\ell} = \sum_{k=1}^{p}\{b_{k\ell}e_k + a_{k\ell}e_{p+k}\} \end{cases} \\ &\Longleftrightarrow T = \begin{pmatrix} a & b & \mathbf{0} \\ -b & a & \mathbf{0} \\ {}^t\mathbf{0} & {}^t\mathbf{0} & 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SO}(2p). \end{aligned}$$

Noticing that

$$\mathrm{U}(p) = \left\{ a + \sqrt{-1}b;\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SO}(2p) \right\},$$

we obtain (ii). We omit the proof of (iii), (iv). □

Remark. The projection $\pi : \mathcal{J}_p \to S^{2p}$ is also obtained as follows. For $V \in \mathcal{J}_p$, we get the orthogonal decomposition with respect to $(\ ,\ )$,

$$\mathbf{C}^{2p+1} = V \oplus \overline{V} \oplus \mathbf{C}Z,$$

where $\overline{V} := \{\overline{v};\ v \in V\}$ and for $Z$ with $(Z, Z) = 1$. Then $\pi(V) = \pm Z \in S^{2p} \subset \mathbf{R}^{2p+1}$.

Now Calabi's theorem says

THEOREM (2.39) (Calabi [C1], [C2]). *Let* $\phi : (S^2, g_{S^2}) \to (S^n, g_{S^n})$ *be a full harmonic mapping. Then*

(i) $n = 2p$ *(an even number),*

*and*

(ii) *there exists a holomorphic mapping* $\psi : P^2(\mathbf{C}) \to \mathcal{J}_p$ *which is horizontal with respect to the Riemannian submersion* $\pi : (\mathcal{J}_p, g') \to (S^{2p}, g_{S^{2p}})$, *and satisfies* $\phi = \pi \circ \psi$.

REMARK. Due to Proposition (2.36), for any horizontal holomorphic mapping $\psi : P^1(\mathbf{C}) \to \mathcal{J}_p$, $\phi = \pi \circ \psi : (S^2, g_{S^2}) \to (S^{2p}, g_{S^{2p}})$ is always harmonic. Theorem (2.39) claims that the converse holds. We recommend Lawson's exposition [Lw]. See also Verdier [V], Loo [Loo], Kotani [Kt], Furuta-Guest-Kotani-Ohnita [F.G.K.O] and Guest-Ohnita [G.O] about the structure of the moduli of all harmonic mappings of $(S^2, g_{S^2})$ into $(S^n, g_{S^n})$.

**2.4. Group equivariant harmonic mappings.** It is quite difficult to solve (2.7) directly, so we consider reducing (2.7) to the ordinary differential equation under the group equivariance condition, and solve or show the existence of a solution of this ODE. To do this, we consider the following situation:

(I) Let $(M, g)$ be a compact Riemannian manifold on which a compact Lie group $K$ acts isometrically and with **cohomogeneity one**, i.e., denoting the action of $K$ on $M$ by

$$K \times M \ni (k, p) \mapsto \tau_k p := k \cdot p \in M,$$

(i) $\tau_k{}^* g = g$, $\forall k \in K$, and

(ii) there exists a geodesic $c(t)$, $0 \le t \le \ell$ such that

$$t = \int_0^t g(c'(s), c'(s))^{1/2}\, ds, \tag{2.40}$$

$$\dim(K\, c(t)) = \dim M - 1 \text{ for all } 0 < t < \ell, \tag{2.41}$$

$$\dim(K\, c(t)) < \dim M - 1 \text{ for } t = 0, \ell,$$

$$M = \bigcup_{t \in [0, \ell]} K\, c(t), \tag{2.42}$$

where $K\, c(t) : \{k \cdot c(t);\ k \in K\}$ is called the **K-orbit**. (See Figure 6.3, next page.)

(II) Assume that the $K$-invariant Riemannian metric $g$ can be expressed as follows: Let $J_t$ be the isotropy subgroup of $K$ at $c(t)$, which satisfies the same group, denoted by $J$, for $0 < t < \ell$. Let $\mathfrak{k}$, $\mathfrak{j}$ be the Lie algebras of $K$, $J$, respectively. Let

$$\mathfrak{k} = \mathfrak{j} \oplus \mathfrak{m},$$

be the orthogonal decomposition with respect to the Ad$(K)$-invariant fixed inner product $(\ ,\ )$ (cf. (4.17) in Chapter 2), and assume the Riemannian metric $g$ satisfies

$$g_{kc(t)}(\tau_{k*} X_{ic(t)}, \tau_{k*} X_{jc(t)}) = f_i(t)^2 \delta_{ij}, \qquad 1 \le i, j \le m - 1 \tag{2.43}$$

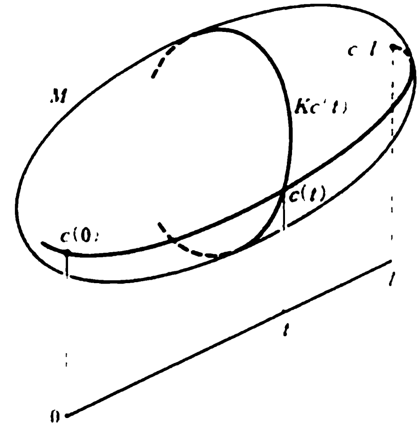



FIGURE 6.3

for $k \in K$, $0 < t < \ell$, and some orthonormal basis $\{X_1, \dots, X_{m-1}\}$ of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$ (see (4.20) in Chapter 2 for the meaning of the left hand side). Here we assume the $f_i(t)$ are positive $C^\infty$-functions on the open interval $(0, \ell)$.

Under the assumptions (I), (II) of $(M, g)$, let us consider a $C^\infty$-mapping $\phi: (M, g) \to (S^n, g_{S^n})$ satisfying the following condition:

(III) Let $A: K \to SO(n+1)$ be a homomorphism, and assume $\phi$ satisfies

$$\phi(k \cdot p) = A(k)\,\phi(p), \qquad k \in K,\ p \in M. \tag{2.44}$$

Such a $\phi$ is called an **A-equivariant mapping**.

Under the above setting, the Euler-Lagrange equation (2.7) can be reduced to an ordinary differential equation.

PROPOSITION (2.45) (cf. [Ur11]). *Let $(M, g)$ be a compact Riemannian manifold satisfying* (I), (II), *and a $C^\infty$-mapping $\phi: (M, g) \to (S^n, g_{S^n})$ satisfying* (III). *Assume, denoting $\phi$ as*

$$\Phi = \iota \circ \phi = (\phi_1, \dots, \phi_{n+1}),$$

*that $\Phi(t) = (\phi_1(t), \dots, \phi_{n+1}(t)) = \Phi(c(t))$, $0 < t < \ell$, where $\iota: S^n \subset \mathbf{R}^{n+1}$ denotes the inclusion. Then a necessary and sufficient condition for $\phi$ to satisfy the Euler-Lagrange equation $\tau(\phi) = 0$ on the open dense subset $M' := \{kc(t);\ k \in K,\ 0 < t < \ell\}$ is that $\Phi(t)$ satisfies the following equation:*

$$-\Phi''(t) - \frac{D'(t)}{D(t)}\Phi'(t) - \sum_{j=1}^{m-1} f_j(t)^{-2} dA(X_j)^2 \Phi(t) \tag{2.46}$$

$$= \left\{ |\Phi'(t)|^2 + \sum_{j=1}^{m-1} f_j(t)^{-2} |dA(X_j)\Phi(t)|^2 \right\} \Phi(t), \qquad 0 < t < \ell,$$

*where* $\Phi'(t) := (\phi_1'(t), \dots, \phi_{n+1}'(t))$, $D(t) := \prod_{j=1}^{m-1} f_j(t)$, *and see* (4.8) *in Chapter* 2 *for the definition of* $dA(X)$, $X \in \mathfrak{m}$.

The calculation (2.46) follows from making use of the $A$-equivariance condition (2.44), and both the $\Delta_g \phi_i$ and $\sum_{j=1}^{m-1} \langle d\phi_j, d\phi_j\rangle$ of (2.7) are equal to the left-hand side, the right-hand side of (2.46), respectively. We omit the calculation.

Now for a solution $\Phi$ of (2.46), we put

$$\phi(kc(t)) := A(k)\Phi(t), \qquad k \in K,\ 0 < t < \ell. \tag{2.47}$$

Then this mapping $\phi$ is a $C^\infty$-mapping on the open dense subset $M'$ of $M$ into $S^n$, and if we may determine the boundary value $\Phi(0)$ and $\Phi(\ell)$ for $\phi$ to be a continuous mapping on the whole $M$ into $S^n$, then we obtain a $C^\infty$-$A$-equivariant harmonic mapping $\phi : (M, g) \to (S^n, g_{S^n})$ due to the following proposition.

PROPOSITION (2.48). *Let* $(M, g)$, $(N, h)$ *be compact Riemannian manifolds, let* $\phi \in L_{1,2}(M, N)$ *be a weak solution of the Euler-Lagrange equation, i.e., a solution of* (1.23) *or* (2.23) *in Chapter* 4 *in the sense of distribution. Assume that* $\phi$ *is a continuous, i.e.,* $\phi \in C^0(M, N)$. *Then* $\phi \in C^\infty(M, N)$ *and* $\phi : (M, g) \to (N, h)$ *is a harmonic mapping.*

For a proof, see Borchers-Garber [B.G] and Schoen [Sc].

Now we give examples making use of Proposition (2.45).

EXAMPLE 1. If $(M, g) = (S^n, g_{S^n})$, let

$$K := SO(n-1) \times SO(2) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} ;\ x \in SO(n-1),\ y \in SO(2) \right\} \subset SO(n+1).$$

Via the inclusion $K \subset SO(n+1)$ and the action of $SO(n+1)$ on $S^n$ in (4.23) Example 3 in Chapter 2, $K$ acts on $S^n$ satisfying the conditions (I) and (II): In fact, let $\{e_1, \dots, e_{n+1}\}$ be the standard basis of $\mathbf{R}^{n+1}$ and let

$$c(t) := \cos t\, e_1 + \sin t\, e_n, \qquad 0 \le t \le 2\pi,$$

be a geodesic of $(S^n, g_{S^n})$. Then the isotropy subgroup of $K$ at $c(t)$ for $0 < t < \frac{\pi}{2}$, is given by

$$J_t = \left\{ \left( \begin{array}{cc|c} 1 & 0 & \\ 0 & X' & 0 \\ \hline & & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \\ 0 & & \end{array} \right) : X' \in SO(n-2) \right\}$$

denoted by $J$. The inner product

$$(X, Y) = -\frac{1}{2}\operatorname{tr}(XY), \qquad X, Y \in \mathfrak{so}(n+1)$$

on $\mathfrak{k} \subset \mathfrak{so}(n+1)$ is the desired inner product due to (4.17) in Chapter 2, and we can take as an orthonormal basis $\{X_1, \dots, X_{n-1}\}$ of the orthogonal complement $\mathfrak{m}$ of $\mathfrak{j}$ in $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$,

$$X_i := i+1 \begin{pmatrix} 0\ 0 \cdots \overset{i+1}{-1} \cdots 0 & \\ 0 & \\ \vdots & \\ 1 \quad\quad 0 & 0 \\ \vdots & \\ 0 & \\ \hline 0 & 0 \end{pmatrix}, \quad 1 \le i \le n-2, \qquad X_{n-1} := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right).$$

Then we get

$$f_1(t) = \cdots = f_{n-2}(t) = \cos t, \qquad f_{n-1}(t) = \sin t;$$

thus,

$$D(t) = (\cos t)^{n-2} \sin t.$$

Then $\dim K/J_t = n-1$, $(0 < t < \frac{\pi}{2})$, and $\ell = \frac{\pi}{2}$, and $(M, g) = (S^n, g_{S^n})$ equiped with the action of $K = SO(n-1) \times SO(2)$ satisfies conditions (I) and (II).

Furthermore, let us consider as a homomorphism $A$ of $K$ into itself,

$$A = A_a : K = SO(n-1) \times SO(2) \ni \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ 0 & y^a \end{pmatrix} \in SO(n+1),$$

for $a \in \mathbf{Z}$. As $\Phi(t)$, we take

$$\Phi(t) = \cos r(t)\, e_1 + \sin r(t)\, e_n, \qquad 0 < t < \frac{\pi}{2},$$

where $r(t)$ is a real-valued function on the open interval $(0, \frac{\pi}{2})$, and define an $A_a$-equivariant mapping $\phi_a$ by

$$\phi_a\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} c(t)\right) = A_a\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \Phi(t).$$

Since

$$dA_a(X_i) = X_i\, (1 \le i \le n-2), \qquad dA_a(X_{n-1}) = aX_{n-1},$$

(2.46) is reduced to the following ordinary differential equation:

$$r'' + (\cos t - (n-2)\tan t) r' + \left(\frac{n-2}{\cos^2 t} - \frac{a^2}{\sin^2 t}\right) \sin r \cos r = 0. \tag{2.49}$$

Furthermore, if $r = r(t)$ satisfies the following condition (2.50), then the above $\phi_a$ becomes a continuous mapping of $S^n$ into itself:

$$0 \le r(t) \le \frac{\pi}{2}\ \left(0 < t < \frac{\pi}{2}\right), \qquad r(0) = 0,\ r\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \tag{2.50}$$

Thus, it should be possible to show the existence of a solution $r = r(t)$ of (2.49) with (2.50). This was studied by W.Y. Ding [Dg], and his results are:

(i) If $3 \le n \le 7$, there exists a solution of (2.49) with (2.50) for all $a \in \mathbf{Z}$.

(ii) If $n \ge 8$, there is no solution of (2.49) with (2.50) except in the case $a = 0, \pm 1$.

Thus, we obtain that if $3 \le n \le 7$, we obtain the existence of a harmonic mapping $\phi_a : (S^n, g_{S^n}) \to (S^n, g_{S^n})$. Note that the mapping degree of $\phi_a$ satisfies $\deg(\phi_a) = a$.

Recall the definition of the mapping degree: In general, $\dim M = \dim N = n$, the **mapping degree** $\deg(\phi)$ of $\phi \in C^\infty(M, N)$ is by definition

$$\deg(\phi) := \int_M f\, v_g / \mathrm{Vol}(N, h), \tag{2.51}$$

for Riemannian metrics $g$, $h$ on $M$, $N$, respectively. Here $\phi^* v_h = f\, v_g$, $f \in C^\infty(M)$, and the mapping degree is independent on the choices of $g$, $h$ and takes an integer value assuming $M$ and $N$ are orientable, i.e., they admit $C^\infty$-$n$-forms that are every where nonzero. Then it is known (see [Ml], [G.L.P]) that

(i) if $\phi_1, \phi_2 \in C^\infty(M, N)$ are homotopic, then

$$\deg(\phi_1) = \deg(\phi_2).$$

(ii) (Hopf's Theorem) If $N = S^n$, then for $\phi_1, \phi_2 \in C^\infty(M, S^n)$,

$$\phi_1 \text{ and } \phi_2 \text{ are homotopic} \iff \deg(\phi_1) = \deg(\phi_2).$$

(iii) If $M = N = S^n$, then $[M, N] = \pi_n(S^n) \cong \mathbf{Z}$ and the correspondance is given by

$$\gamma = [\phi] \mapsto \deg(\phi).$$

To show $\deg(\phi_a) = a$, we have only to calculate (2.51) and then

$$\mathrm{Vol}(S^n, g_{S^n}) = \int_0^{\pi/2} \cos t\,(\sin t)^{n-2}\, dt,$$

and also

$$\begin{aligned}\int_{S^n} \phi_a^* v_{g_{S^n}} &= a \int_0^{\pi/2} \dot{r} \cos r\,(\sin r)^{n-2}\, dt \\ &= a \int_{r(0)}^{r(\pi/2)} \cos r\,(\sin r)^{n-2}\, dr = a\, \mathrm{Vol}(S^n, g_{S^n})\end{aligned}$$

which yields $\deg(\phi_a) = a$.

If $n = 2$, the holomorphic mapping $\mathbf{C} \ni z \mapsto z^a \in \mathbf{C}$ induces a harmonic mapping $\phi_a : S^2 \to S^2$ with $\deg(\phi_a) = a$, regarding as $S^2 = \mathbf{C} \cup \{\infty\}$.

If $n = 1$, the mapping $S^1 \ni e^{i\theta} \mapsto e^{ia\theta} \in S^1$ is a harmonic mapping with $\deg(\phi_a) = a$. Thus, we obtain

THEOREM (2.52) (R. T. Smith [St 2], 1975). *If* $1 \leq n \leq 7$, *then each element* $\pi_n(S^n) \cong \mathbf{Z}$ *can be represented by a harmonic mapping of* $(S^n, g_{S^n})$ *into itself.*

EXAMPLE 2. In the case $(M, g) = (\mathbf{R}^2/\Lambda, g_\Lambda)$, where $\Lambda = 2\pi\mathbf{Z} \times T\mathbf{Z}$, we put

$$K = \mathbf{R}/2\pi\mathbf{Z} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ;\ \theta \in \mathbf{R} \right\}$$

of which action on $\mathbf{R}^2/\Lambda$ is

$$K \times \mathbf{R}^2/\Lambda \ni ([\theta'], \pi(\theta, t)) \mapsto \pi(\theta + \theta', t) \in \mathbf{R}^2/\Lambda.$$

The constant $T$ will be fixed as follows. Given a geodesic of $(\mathbf{R}^2/\Lambda, g_\Lambda)$, $c(t) = \pi(0, t)$, $0 \leq t \leq T$, the isotropy subgroup of $K$ at $c(t)$, $J_t = \{e\}$, consider the following two homomorphisms:

(i) For $a \in \mathbf{Z}$,

$$A_a : K \in x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x^a \end{pmatrix} \in SO(3).$$

(ii) For $a, b \in \mathbf{Z}$,

$$A_{a,b} : K \ni x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} x^a & 0 \\ 0 & x^b \end{pmatrix} \in SO(4).$$

For (i), let

$$\Phi(t) := \cos r(t)\, e_1 + \sin r(t)\, e_2 \in S^2,$$
$$\phi(t, \theta) := \cos r(t)\, e_1 + \sin r(t)(\cos a\theta\, e_2 + \sin a\theta\, e_3) \in S^2,$$

for (ii), let

$$\Phi(t) := \cos r(t)\, e_1 + \sin r(t)\, e_3 \in S^3,$$
$$\phi(t, \theta) := \cos a\theta\, e_1 + \sin a\theta\, e_2 + \sin r(t)(\cos b\theta\, e_3 + \sin b\theta\, e_4) \in S^3.$$

Then the differential equation for these $\phi$ to be harmonic is

$$r'' - a^2 \sin r \cos r = 0, \tag{2.53}$$
$$r'' + (a^2 - b^2) \sin r \cos r = 0, \tag{2.54}$$

for (i) and (ii), respectively.

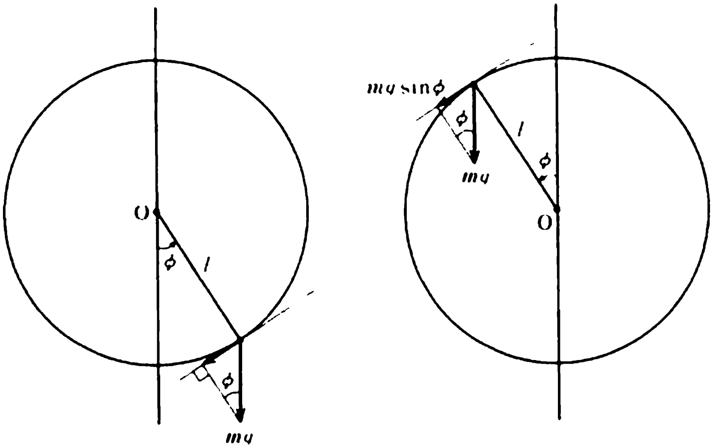
For the equivariant mapping to the solutions $r = r(t)$ of (2.53), (2.54) to be defined on all of $\mathbf{R}^2/\Lambda$, it suffices for $(\cos r(t), \sin r(t))$ to be periodic in $t$:

$$(\cos r(t + T), \sin r(t + T)) = (\cos, \sin r(t)),$$

for all $t$. This period $T > 0$ should be the $T$ used to define $\Lambda$. Moreover, if we put $\phi(t) := 2r(t)$, then both (2.53) and (2.54) become to be the equation of the pendulum:

$$\phi'' - k \sin\phi = 0,$$

where $k = a^2$, $k = -(a^2 - b^2)$ for (2.53), (2.54), respectively.




(i) $ml\phi'' = -mg\sin\phi$ (ii) $ml\phi'' = mg\sin\phi$

FIGURE 6.4

(2.56) *The pendulum.* Here we consider the equation for the swing of a pendulum of mass $m$ and string length $\ell$. Denote by $g$ the gravitational constant.

(i) In the case (i) of Figure 6.4, the strength of force by the motion of angle speed $\phi$ is $m\ell\phi''$, and the force of gravitation is $mg \sin\phi$ in the oposite direction, and then to balance them, the equation should be

$$m\ell\phi'' = -mg \sin \phi.$$

Thus, we obtain

$$\phi'' + \frac{g}{\ell} \sin \phi = 0.$$

(ii) The equation for case (ii) is

$$\phi'' - \frac{g}{\ell} \sin \phi = 0.$$

LEMMA (2.57). *For a solution* $\phi$ *of* (2.55),

$$E := \frac{1}{2}\phi'^2 + k \cos \phi$$

*is a constant independent in* $t$.

PROOF. The proof is clear from

$$E' = \phi'\phi'' - k\phi' \sin \phi = \phi'(\phi'' - k \sin \phi) = 0. \quad \square$$

Lemma (2.57) has the following implication: The sum of the kinetic energy and the potential energy coincides with

$$\frac{1}{2}m(\ell\phi')^2 + mg\ell \cos \phi = \text{constant}. \tag{2.58}$$

That is, the total energy of the pendulum is constant.

On the other hand, we want solutions such that $(\cos\phi(t), \sin\phi(t))$ is periodic in $t$. However, note that the position of the pendulum is given by

$$(\ell\cos\phi, -\ell\sin\phi) \quad \text{or} \quad (-\ell\cos\phi, \ell\sin\phi). \tag{2.59}$$

Since the total energy of the pendulum is constant in time $t$, there are many solutions for which positions are periodic in $t$. Thus, we obtain

PROPOSITION (2.60) (cf. [St 2], [Ur 11]). *A solution of the equation of the pendulum gives a harmonic mapping of a 2-dimensional flat torus* $(\mathbf{R}^2/\Lambda, g_\Lambda)$ *into* $(S^2, g_{S^2})$ *or* $(S^3, g_{S^3})$.

## §3. The case of symmetric spaces

**3.1. Case of complex projective spaces.** Let $(P^n(\mathbf{C}), h)$ be the complex projective space, and let $h$ be an $\mathrm{SU}(n+1)$-invariant Kähler metric on it (cf. (3.17) in Example 9 in Chapter 4). Then any harmonic mapping of $(S^2, g_{S^2})$ into $(P^n(\mathbf{C}), h)$ can be described by a holomorphic mapping as follows. Here a harmonic or holomorphic mapping $\phi : (S^2, g_{S^2}) \to (P^n(\mathbf{C}), h)$ is said to be full if the image $\phi(S^2)$ is contained in no $(n-1)$-dimensional projective subspace $P^{n-1}(\mathbf{C})$ of $P^n(\mathbf{C})$. Then

THEOREM (3.1) (A. M. Din-W. J. Zakrzewski [D.Z], V. Glaser-R. Stora [G.S], J. Eells-J. C. Wood [E.W]). *There is a one-to-one correspondance between the set of all full harmonic mappings* $\phi : (S^2, g_{S^2}) \to (P^n(\mathbf{C}), h)$ *and*

$$\{(f, r);\ f \text{ is full holomorphic of } P^1(\mathbf{C}) \text{ into } P^n(\mathbf{C}),\ r = 0, 1, \dots, n\}.$$

The correspondance in Theorem (3.1) is given as follows.

Let $\pi : \mathbf{C}^{n+1} - (0) \ni z \mapsto [z] \in P^n(\mathbf{C})$ be the canonical projection, and let $w$ be a complex coordinate on an open subset in $S^2 = P^1(\mathbf{C})$. Now let $f : P^1(\mathbf{C}) = S^2 \to P^n(\mathbf{C})$ be a full holomorphic mapping and let $0 \le r \le n$. Then we may choose a holomorphic mapping

$$F : U \to \mathbf{C} - (0) \quad \text{with } f = \pi \circ F.$$

We denote $F(w) = (F_1(w), \dots, F_{n+1}(w))$, $w \in U$, and the $i$th differentiation of $F$, $1 \le i \le n$, is written as

$$\frac{\partial^i F}{\partial w^i}(w) := \left(\frac{\partial^i F_1}{\partial w^i}(w), \dots, \frac{\partial^i F_{n+1}}{\partial w^i}(w)\right) \in \mathbf{C}^{n+1}, \qquad w \in U.$$

For any $\ell = 0, 1, \dots, n$, we denote by $F_\ell(w)$ the complex subspace of $\mathbf{C}^{n+1}$ generated by

$$\left\{F(w), \frac{\partial F}{\partial w}(w), \dots, \frac{\partial^\ell F}{\partial w^\ell}(w)\right\}.$$

Then since $f$ is full, we get

$$\dim_{\mathbf{C}} F_\ell(w) = \ell + 1, \qquad \ell = 0, 1, \dots, n.$$

This $F_\ell$ gives a holomorphic mapping of $U$ into the Grassmann manifold,

$$F_\ell : S^2 \supset U \ni w \mapsto F_\ell(w) \in G_{\ell+1}(\mathbf{C}^{n+1}),$$

Here $G_{\ell+1}(\mathbf{C}^{n+1})$ is a compact complex manifold consisting of all $(\ell+1)$-dimensional complex subspaces of $\mathbf{C}^{n+1}$, called a Grassmannian manifold.

Again using the assumption of fullness of $f$, it turns out that $F_\ell$ can be extended to a holomorphic mapping of the whole $S^2$ into $G_{\ell+1}(\mathbf{C}^{n+1})$.

For all $0 \le r \le n$, we define

$$\mathcal{F}_r := \{(U, V) \in G_r(\mathbf{C}^{n+1}) \times G_{r+1}(\mathbf{C}^{n+1});\ U \subset V\}.$$

$\mathcal{F}_r$ is a compact complex manifold and can be expressed as a homogeneous space

$$\mathcal{F}_r = U(n+1)/(U(1) \times U(r) \times U(n-r))$$

which admits a $U(n+1)$-invariant Riemannian metric $g'$ such that

(i) $(\mathcal{F}_r, g')$ is a Kähler manifold, and

(ii) the canonical projection $\pi_r$ coming from

$$P^n(\mathrm{C}) = \mathrm{SU}(n+1)/S(\mathrm{U}(1) \times \mathrm{U}(n)) = \mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n)),$$

gives a Riemannian submersion $\pi_r : (\mathcal{F}_r, g') \to (P^n(\mathrm{C}), h)$ (cf. (3.7) in Chapter 7).

Note that $\pi_r$ is given by

$$\pi_r : \mathcal{F}_r \ni (U, V) \mapsto V \ominus U \in P^n(\mathrm{C}),$$

where $V \ominus U$ is the orthogonal complement of $U$ in $V$, and $\pi_r$ is not holomorphic.

Consider the holomorphic mapping

$$\Phi_r := (F_{r-1}, F_r) : S^2 = P^1(\mathrm{C}) \ni w \mapsto (F_{r-1}(w), F_r(w)) \in \mathcal{F}_r$$

which is harmonic due to (3.14) in Chapter 4. Moreover, it turns out that $\Phi_r$ is horizontal with respect to the Riemannian submersion $\pi_r : (\mathcal{F}_r, g') \to (P^n(\mathrm{C}), h)$ (cf. (ii) (2.36)). Thus, due to Proposition (2.36),

$$\phi_r := \pi_r \circ \Phi_r$$

is a harmonic mapping of $(S^2, g_{S^2})$ into $(P^n(\mathrm{C}), h)$.

Theorem (3.1) asserts that the converse of the above is true, that is, there exists an inverse correspondance of the above.

**3.2. The unitary group and chiral fields.** Here we show there exists a one-to-one correspondance between the set of all harmonic mappings of $(S^2, g_{S^2})$ into the unitary group $\mathrm{U}(n)$ and the set of Yang-Mills connections (then flat connections), called **chiral fields** with certain asymptotic conditions of the trivial vector bundle over $\mathbf{R}^2$ with the structure group $\mathrm{U}(n)$, $E = \mathbf{R}^2 \times \mathbf{C}^n$.

(3.2) *The unitary group.* Recall from subsection 4.2 in Chapter 4 that

$$\mathrm{U}(n) := \{a \in M(n, \mathbf{C});\ {}^t\bar{a}\,a = a\,{}^t\bar{a} = I\},$$
(the unitary group of degree $n$),
$$\mathrm{u}(n) := \{X \in M(n, \mathbf{C});\ {}^t\overline{X} + X = 0\}, \quad \text{(the Lie algebra of } \mathrm{U}(n)).$$

Define the inner product $\langle\ ,\ \rangle$ and the norm $|\ |$ on $\mathrm{u}(n)$ by

$$\langle X, Y\rangle := -\operatorname{tr}(XY), \qquad X, Y \in \mathrm{u}(n),$$
$$|X|^2 := \langle X, X\rangle,$$

and the biinvariant Riemannian metric $h$ on $U(n)$ by

$$h_a(X_a, Y_a) = \langle X, Y\rangle, \qquad X, Y \in \mathrm{u}(n),\ a \in U(n).$$

We denote the standard Riemannian metric on $\mathbf{R}^2$ by

$$g_0 = dx \otimes dx + dy \otimes dy,$$

where $(x, y)$ is the standard coordinate on $\mathbf{R}^2$. Let $\Omega \subset \mathbf{R}^2$ any domain in $\mathbf{R}^2$. For a $C^\infty$-mapping $\phi : \Omega \to U(n)$, the energy $E(\phi)$ is given by

$$E(\phi) := \frac{1}{2}\int_\Omega \left(\left|\phi^{-1}\frac{\partial\phi}{\partial x}\right|^2 + \left|\phi^{-1}\frac{\partial\phi}{\partial y}\right|^2\right) dx\,dy, \tag{3.3}$$

as we show below. Here for $\phi = (\phi_{ij})$ (a unitary matrix of degree $n$), we define the matrices of degree $n$

$$\frac{\partial\phi}{\partial x} := \left(\frac{\partial\phi_{ij}}{\partial x}\right), \qquad \frac{\partial\phi}{\partial y} := \left(\frac{\partial\phi_{ij}}{\partial y}\right),$$

then

$$\phi^{-1}\frac{\partial\phi}{\partial x}, \qquad \phi^{-1}\frac{\partial\phi}{\partial y} \in \mathrm{u}(n). \tag{3.4}$$

The norm $|\cdot|$ in the integrand of (3.3) is the norm of $\mathrm{u}(n)$. Then we obtain

(3.5) The energy density function $e(\phi)$ of a $C^\infty$ mapping $\phi : (\Omega, g_0) \to (U(n), h)$ is given by

$$e(\phi) = \frac{1}{2}\left(\left|\phi^{-1}\frac{\partial\phi}{\partial x}\right|^2 + \left|\phi^{-1}\frac{\partial\phi}{\partial y}\right|^2\right).$$

Proof. In fact, for $p \in \Omega$,

$$h\left(\phi_*\left(\frac{\partial}{\partial x}\right)_p, \phi_*\left(\frac{\partial}{\partial x}\right)_p\right) = h\left(L_{\phi(p)^{-1}*}\phi_*\left(\frac{\partial}{\partial x}\right)_p, L_{\phi(p)^{-1}*}\phi_*\left(\frac{\partial}{\partial x}\right)_p\right)$$
$$= \left|\phi(p)^{-1}\frac{\partial\phi}{\partial x}(p)\right|^2,$$

and the same is true for $\frac{\partial}{\partial y}$. □

(3.6) A sufficient and necessary condition for $\phi : (\Omega, g_0) \to (U(n), h)$ to be a harmonic mapping is

$$\frac{\partial}{\partial x}\left(\phi^{-1}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi^{-1}\frac{\partial \phi}{\partial y}\right) = 0 \quad \text{on } \Omega.$$

PROOF. Let $\phi_t$ be a variation of $\phi$ with $\phi_0 = \phi$, and set

$$\Lambda := \left.\frac{d}{dt}\right|_{t=0} \phi^{-1}\phi_t,$$

then $\Lambda : \Omega \to \mathrm{u}(n)$. We can assume that

$$\Lambda = 0 \quad \text{on } \partial\Omega.$$

Then we obtain

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = \int_\Omega \left\{\left\langle \left.\frac{d}{dt}\right|_{t=0} \phi_t^{-1}\frac{\partial \phi_t}{\partial x}, \phi^{-1}\frac{\partial \phi}{\partial x}\right\rangle + \left\langle \left.\frac{d}{dt}\right|_{t=0} \phi_t^{-1}\frac{\partial \phi_t}{\partial y}, \phi^{-1}\frac{\partial}{\partial y}\right\rangle\right\} dx\, dy, \tag{$*$}$$

here we get

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^{-1}\frac{\partial \phi_t}{\partial x} = \left.\frac{d}{dt}\right|_{t=0} \left\{(\phi^{-1}\phi_t)^{-1}\phi^{-1}\frac{\partial \phi_t}{\partial x}\right\}$$
$$= -\Lambda\phi^{-1}\frac{\partial \phi}{\partial x} + \frac{\partial \Lambda}{\partial x} + \phi^{-1}\frac{\partial \phi}{\partial x}\Lambda.$$

Therefore, denoting $B := \phi^{-1}\frac{\partial \phi}{\partial x}$, we get

$$\left\langle \left.\frac{d}{dt}\right|_{t=0} \phi_t^{-1}\frac{\partial \phi_t}{\partial x}, B\right\rangle = -(\Lambda B, B) + \left\langle \frac{\partial \Lambda}{\partial x}, B\right\rangle + (B\Lambda, B)$$
$$= \left\langle \frac{\partial \Lambda}{\partial x}, B\right\rangle.$$

Because

$$(\Lambda B, B) = -\operatorname{tr}(\Lambda BB) = -\operatorname{tr}(B\Lambda B) = (B\Lambda, B).$$

In the same way, we can calculate $\langle \frac{d}{dt}|_{t=0}\phi_t^{-1}\frac{\partial \phi_t}{\partial y}, \phi^{-1}\frac{\partial \phi}{\partial y}\rangle$, and then we obtain ($*$) as

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = \int_\Omega \left\{\left\langle \frac{\partial \Lambda}{\partial x}, \phi^{-1}\frac{\partial \phi}{\partial x}\right\rangle + \left\langle \frac{\partial \Lambda}{\partial y}, \phi^{-1}\frac{\partial \phi}{\partial y}\right\rangle\right\} dx\, dy$$
$$= \int_\Omega \left\langle \Lambda, \frac{\partial}{\partial x}\left(\phi^{-1}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi^{-1}\frac{\partial}{\partial y}\right)\right\rangle dx\, dy$$

which follows from Stokes' theorem (subsection 3.3 in Chapter 1). Here $\Lambda$ is arbitrary; we obtain the desired. □

(3.7) The energy $E(\phi)$ for a Riemannian metric $g := \lambda^2 g_0$, $\lambda \in C^\infty(\Omega)$, $\lambda > 0$ on $\Omega$, of a mapping

$$\phi : (\Omega, g) \to (U(n), h)$$

coincides with (3.3) (cf. exercise 4.2), and it follows that

$$\begin{aligned}&\phi : (\Omega, g) \to (U(n)) \text{ is harmonic}\\ &\iff \phi : (\Omega, g_0) \to (U(n), h) \text{ is harmonic}\\ &\iff \text{the equation in (3.6) holds.}\end{aligned}$$

Now define

$$A_x := \phi^{-1}\frac{\partial\phi}{\partial x}, \qquad A_y := \phi^{-1}\frac{\partial\phi}{\partial y}. \tag{3.8}$$

Then $A_x$, $A_y$ are u$(n)$-valued functions on $\Omega$, and the equation in (3.6) is equivalent to

$$\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y = 0 \quad \text{on } \Omega.$$

Next regard (3.8) as a system of differential equations with an unknown function $\phi$, then we get

$$\frac{\partial\phi}{\partial x} = \phi A_x, \qquad \frac{\partial\phi}{\partial y} = \phi A_y, \tag{3.8$'$}$$

and the integrability condition of (3.8′), $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ becomes

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x, A_y] = 0, \tag{3.9}$$

where [ , ] means the commutator of matrices of degree $n$. Thus, we obtain

PROPOSITION (3.10). *The problem of finding a harmonic mapping $\phi : (\Omega, g) \to (U(n), h)$, is reduced to determining two* u$(n)$*-valued functions $A_x$, $A_y$ on $\Omega$ satisfying*

$$\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y = 0, \tag{3.6}$$

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x, A_y] = 0. \tag{3.9}$$

*Given such $A_x$, $A_y$, there exists a unique solution $\phi$ of (3.8′) on $\Omega$ with $\phi(x_0, y_0) = \phi_0$ for every $(x_0, y_0) \in \Omega$, $\phi_0 \in U(n)$ such that $\phi : (\Omega, g) \to (U(n), h)$ is harmonic. ($\{A_x, A_y\}$ is called a* **chiral field**.)

(3.11) Consider the trivial bundle (the product bundle) with structure group U$(n)$ on $\Omega$,

$$E = \Omega \times \mathbf{C}^n.$$

Then the space $\Gamma(E)$ of all $C^\infty$-sections of $E$ is the set of all $\mathbf{C}^n$-valued $C^\infty$-functions on $\Omega$. Connections of $E$ are given as follows: For a chiral field $\{A_x, A_y\}$ on $\Omega$, define $A := A_x dx + A_y dy$. This is a u$(n)$-valued 1-form on $\Omega$, so we can define a connection ${}^A\nabla$ on $E$ by

$${}^A\nabla_X\sigma := X\sigma + A(X)\sigma = d\sigma(X) + A(X)\sigma, \qquad X \in \mathfrak{X}(\Omega) \tag{3.12}$$

for $\sigma \in \Gamma(E)$. Here $(A(X)\sigma)(p) := A(X)(\sigma(p))$, $p \in \Omega$ the right-hand side of which is the multiplication of a matrix and a vector since $A(X) \in \mathfrak{u}(n)$, $\sigma(p) \in \mathbf{C}^n$. Equation (3.12) can be written as

$$^A\nabla = d + A. \tag{3.12$'$}$$

For $A = 0$, we write $\overset{\circ}{\nabla} := d$. Then we obtain

**Proposition 3.13.** (i) *It holds that:*

$$\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y = 0 \Longleftrightarrow \delta^{\overset{\circ}{\nabla}} A = 0.$$

(ii) *Moreover, we get*

$$R^{^A\nabla} = 0, \tag{3.9$'$}$$

*where* $\delta^{\overset{\circ}{\nabla}} A$ *is the codifferential of a* $\mathfrak{u}(n)$*-valued* 1*-form* $A$ *with respect to* $\overset{\circ}{\nabla}$, *and* $R^{^A\nabla}$ *is the curvature tensor of* $^A\nabla$ *(cf. subsection 2.2 of Chapter 5).*

**Proof.** Let $\nabla$ be the Levi-Civita connection of $(\Omega, g)$, and let $\{e_i\}_{i=1}^2$ be an orthonormal frame field with respect to $g = \lambda^2 g_0$. Then we obtain

$$\begin{aligned}
\delta^{\overset{\circ}{\nabla}} A &= -\sum_{i=1}^{2} (\overset{\circ}{\nabla}_{e_i} A)(e_i) \\
&= -\sum_{i=1}^{2} \{\overset{\circ}{\nabla}_{e_i}(A(e_i)) - A(\nabla_{e_i} e_i)\} \\
&= -\sum_{i=1}^{2} \{e_i(A(e_i)) - A(\nabla_{e_i} e_i)\}.
\end{aligned}$$

Here since $e_1 = \frac{1}{\lambda}\frac{\partial}{\partial x}$, $e_2 = \frac{1}{\lambda}\frac{\partial}{\partial y}$,

$$e_1(A(e_1)) = \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}A_x\right), \qquad e_2(A(e_2)) = \frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}A_y\right).$$

On the other hand, since

$$(g_{ij}) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^{-2} \end{pmatrix},$$

we calculate

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= \Gamma^1_{11}\frac{\partial}{\partial x} + \Gamma^2_{11}\frac{\partial}{\partial y} = \lambda^{-1}\left(\frac{\partial\lambda}{\partial x}\frac{\partial}{\partial x} - \frac{\partial\lambda}{\partial y}\frac{\partial}{\partial y}\right), \\
\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= \Gamma^1_{22}\frac{\partial}{\partial x} + \Gamma^2_{22}\frac{\partial}{\partial y} = -\lambda^{-1}\left(\frac{\partial\lambda}{\partial x}\frac{\partial}{\partial x} - \frac{\partial\lambda}{\partial y}\frac{\partial}{\partial y}\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\right)\frac{\partial}{\partial x} + \frac{1}{\lambda^2}\lambda^{-1}\left(\frac{\partial\lambda}{\partial x}\frac{\partial}{\partial x} - \frac{\partial\lambda}{\partial y}\frac{\partial}{\partial y}\right), \\
\nabla_{e_2} e_2 &= \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\right)\frac{\partial}{\partial x} - \frac{1}{\lambda^2}\lambda^{-1}\left(\frac{\partial\lambda}{\partial x}\frac{\partial}{\partial x} - \frac{\partial\lambda}{\partial y}\frac{\partial}{\partial y}\right).
\end{aligned}$$

Thus, we obtain

$$\delta^{\dot\nabla} A = -\frac{1}{\lambda^2}\left\{\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right\}$$

which yields (i).

(ii) Since ${}^A\nabla = \dot\nabla + A$,

$$R^{{}^A\nabla} = R^{\dot\nabla} + d^{\dot\nabla}A + [A, A].$$

Here for $X$, $Y \in \mathfrak{X}(\Omega)$,

$$[A, A](X, Y) = [A(x), A(Y)].$$

For $\sigma \in \Gamma(E)$, we get

$$\begin{aligned} R^{\dot\nabla}\sigma &= \dot\nabla_X(\dot\nabla_Y\sigma) - \dot\nabla_Y(\dot\nabla_X\sigma) - \dot\nabla_{[X,Y]}\sigma \\ &= X(Y\sigma) - Y(X\sigma) - [X, Y]\sigma = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(d^{\dot\nabla}A + [A, A])\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \\ &= \left\{\frac{\partial}{\partial x}\left(A\left(\frac{\partial}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(A\left(\frac{\partial}{\partial x}\right)\right) - A\left(\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\right)\right\} \\ &\quad + \left[A\left(\frac{\partial}{\partial x}\right), A\left(\frac{\partial}{\partial y}\right)\right] \\ &= \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x, A_y] \end{aligned}$$

which yields (ii) due to (3.9). □

Note that for any connection $\nabla$ on $E$, $A := \nabla - \dot\nabla$ is a $\mathfrak{u}(n)$-valued 1-form on $\Omega$ by the definition of connection. By Propositions (3.10), (3.13), we obtain

**Theorem (3.14).** *There exists a one-to-one correspondance between the set of all harmonic mappings of* $(\Omega, g_0)$ *into* $(U(n), h)$ *with*

$$E(\phi) = \int_\Omega e(\phi)\,dx\,dy = \frac{1}{2}\int_\Omega \left\{\left|\phi^{-1}\frac{\partial\phi}{\partial x}\right|^2 + \left|\phi^{-1}\frac{\partial}{\partial y}\right|^2\right\} dx\,dy < \infty$$

*and the set of all Yang-Mills connections* ${}^A\nabla = \dot\nabla + A$ *on the trivial bundle* $E = \Omega \times \mathbf{C}^n$ *with* $\delta^{\dot\nabla}A = 0$ *and*

$$\int_\Omega |A|^2\,dx\,dy < \infty.$$

If $\Omega = \mathbf{R}^2$, it is known that

THEOREM (3.15) (Sacks-Uhlenbeck [S.Uh]). *A necessary and sufficient condition for any harmonic mapping* $\phi : (\mathbf{R}^2, g_0) \to (N, h)$ *to be extended to a unique harmonic one* $\tilde{\phi} : (S^2, g_{S^2}) \to (N, h)$ *is*

$$E(\phi) = \int_{\mathbf{R}^2} e(\phi)\, dx\, dy < \infty.$$

*Any harmonic mapping* $\tilde{\phi} : (S^2, g_{S^2}) \to (N, h)$ *can be obtained in this way.*

Along with this theorem, we obtain

THEOREM (3.16). *There exists a one-to-one correspondance between the set of all harmonic mappings* $\tilde{\phi} : (S^2, g_{S^2}) \to (U(n), h)$ *and the set of all flat connections (Yang-Mills ones, themselves) on the trivial bundle* $E = \mathbf{R}^2 \times \mathbf{C}^n$, ${}^A\nabla = \dot{\nabla} + A$ *with* $\delta^{\nabla} A = 0$ *and*

$$\int_{\mathbf{R}^2} |A|^2\, dx\, dy < \infty.$$

REMARK. Two connections $\nabla$, $\nabla'$ on $E = \mathbf{R}^2 \times \mathbf{C}^n$ are called **gauge equivalent**, denoted $\nabla \sim \nabla'$, if there exists a $\phi \in C^\infty(\mathbf{R}^2, U(n))$ such that $\nabla = \nabla'^{\phi}$ where

$$\nabla'^{\phi} := \phi^{-1} \circ \nabla' \circ \phi = \nabla' + \phi^{-1} d\phi.$$

On the other hand, since ${}^A\nabla = \dot{\nabla} + A$ and $A = A_x dx + A_y dy$, we obtain

$$\begin{aligned} {}^A\nabla \sim \dot{\nabla} &\iff A = \phi^{-1} d\phi, \qquad \phi \in C^\infty(\mathbf{R}^2, U(n)) \\ &\iff A_x = \phi^{-1}\frac{\partial \phi}{\partial x},\ A_y = \phi^{-1}\frac{\partial \phi}{\partial y}, \qquad \phi \in C^\infty(\mathbf{R}^2, U(n)). \end{aligned}$$

See [Uh 5] for more details about harmonic mappings of $(S^2, g_{S^2})$ into $(U(n), h)$.

## §4. Proof of the Eells-Sampson theorem via the variational method

In this section, we give an outline of the the proof due to Uhlenbeck of Eells-Sampson Theorem (1.2) using the method of variations. Recall

THEOREM (1.2) (Eells-Sampson). *Let* $(M, g)$, $(N, h)$ *be compact Riemannian manifolds, and assume that the curvaure of* $(N, h)$ *is nonpositive. Then each homotopy class in* $C^\infty(M, N)$ *can be represented by a harmonic mapping which minimizes the energy in its homotopy class.*

In the following, we assume $h = \iota^* g_0$, where $\iota : N \subset \mathbf{R}^K$ is the inclusion as a closed submanifold of $\mathbf{R}^K$, $g_0$ is the standard Riemannian metric on $\mathbf{R}^K$, and we use the notation in (1.4) and §2 in Chapter 4. Then

THEOREM (4.1) (K.Uhlenbeck [Uh 1], [Uh 2]). *For* $\phi \in C^\infty(M, N)$ *and* $\epsilon > 0$, *let us denote*

$$E(\phi) := \int_M |d\phi|^2 v_g, \qquad J(\phi) := \int_M |d\phi|^{2p} v_g,$$
$$E_\epsilon(\phi) := E(\phi) + \epsilon J(\phi).$$

*Then* $E_\epsilon$ *is defined on* $L_{1,2p}(M, N)$, *and we obtain*

(i) *if* $1 > \frac{m}{2p}$, $m = \dim M$, *then* $E_\epsilon$ *is a* $C^2$*-function on* $L_{1,2p}(M, N)$ *and satisfies the Palais-Smale condition* (C).

(ii) *If* $1 > \frac{m}{2p}$ *and* $(N, h)$ *has nonpositive curvature, then any critical point of* $E_\epsilon$ *on* $L_{1,2p}(M, N)$ *belongs to* $C^\infty(M, N)$.

(iii) *Under the situation of* (ii), *let* $\phi_\epsilon$, $\epsilon > 0$, *be a critical point of* $E_\epsilon$. *Then the set* $\{\phi_\epsilon ; \epsilon > 0\}$ *has an accumulation point* $\phi_0$ *which minimizes the energy in its homotopy class.*

(4.2) *Uhlenbeck's idea of the proof.* We explain the idea of proof of Theorem (4.1). See Figure 6.5. Regard $L_{1,2p}(M, N)$ as $\mathbf{R}^+ := (0, \infty)$, and $E$, $J$, $E_\epsilon$ real-valued functions on it. Let $E$ be a real-valued function on the open interval $(0, \infty)$ defined by

$$E(x) := \frac{1}{x^2} - \frac{1}{x}, \qquad 0 < x < \infty.$$

Then $E$ satisfies

(i) the only critical point of $E$, i.e., $E'(x) = 0$ is $x = 2$ which attains the minimum $-\frac{1}{4}$.

(ii) But since $\lim_{x\to\infty} |E'(x)| = 0$, putting $S := [3, \infty)$,

$$\begin{cases} \inf_S |E'(x)| = 0, \quad \sup_S |E(x)| \le \frac{1}{4}, \quad \text{and} \\ \text{there is no point } x \text{ such that } E'(x) = 0 \text{ on } \overline{S} = [3, \infty). \end{cases}$$

Therefore, $E$ does not satisfy the condition (C).

(iii) However, consider

$$J(x) := x,$$
$$E_\epsilon(x) := E(x) + \epsilon J(x) = \frac{1}{x^2} - \frac{1}{x} + \epsilon x, \qquad \epsilon > 0.$$

Then

$$E_\epsilon'(x) = \frac{\epsilon x^3 + x - 2}{x^3},$$

and $\lim_{x\to\infty} E_\epsilon'(x) = \epsilon$, $E_\epsilon$ satisfies the condition (C) and attains the minimum at $x_\epsilon$ which satisfies $\epsilon x^3 + x - 2 = 0$.

(iv) Thus, as $\epsilon \to 0$, $x_\epsilon$ converges to $x = 2$.

In this way, we can capture the minimizer $x = 2$ of $E$, even $E$ does not satisfy the condition (C). This method is not adequate to find the minimizer of a function on a finite-dimensional space, but becomes a strong method to search for a minimizer of a function on an infinite dimensional space.

Now let us begin a proof of Theorem (4.1).

(i) The proof of (i) in Theorem (4.1) may be carried out by the same argument as in the proof of Theorem (3.3) in Chapter 3, so we omit it.

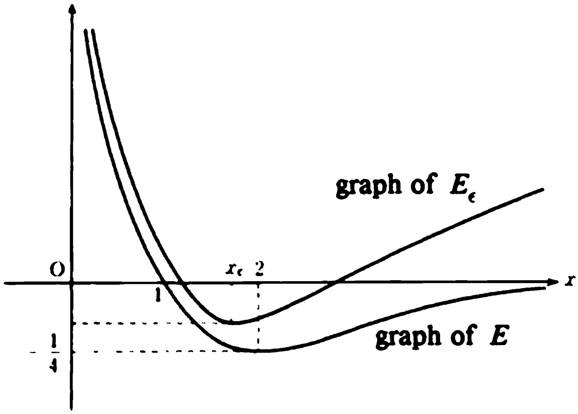



FIGURE 6.5

(ii) To show "Any critical point of $E_\epsilon$ on $L_{1,2p}(M, N)$ is $C^\infty$" in (ii) Theorem (4.1), we first find the Euler-Lagrange equation for $E_\epsilon$ by applying the method in the proof of Theorem (2.22) in Chapter 4 to $E_\epsilon$.

PROPOSITION (4.3). *The Euler-Lagrange equation for* $E_\epsilon = E + \epsilon J$ *is given by*

$$\delta((1 + p\epsilon|d\phi|^{2p-2})d\phi) + (1 + p\epsilon|d\phi|^{2p-2}) \sum_{i,j=1}^{m} A_{\phi(x)}\left(d\phi\left(\frac{\partial}{\partial x_i}\right), d\phi\left(\frac{\partial}{\partial x_j}\right)\right) g^{ij} = 0. \tag{4.4}$$

*Here the first term on the left-hand side is the codifferentiation of a* 1*-form which is given by*

$$\delta((1 + p\epsilon|d\phi|^{2p-2})d\phi) = -\frac{1}{\sqrt{g}} \sum_{i=1}^{m} dX_i\left(\frac{\partial}{\partial x_i}\right),$$

$$X_i = (X_{i1}, \dots, X_{iK}),$$

$$X_{iA} = \sum_{j=1}^{m} \sqrt{g} g^{ij}(1 + p\epsilon|d\phi|^{2p-2})\, d\phi_A\left(\frac{\partial}{\partial x_j}\right), \qquad 1 \le A \le K.$$

Equation (4.4) makes sense for $\phi \in L_{1,2p}(M, N)$ as a distribution solution, called a **weak solution**. The assertion (ii) in Theorem (4.1) means that we can choose a weak solution of (4.4) as a $C^\infty$ solution. In the course of the proof of (ii), the following lemma is essential. The proof can be carried out by hard estimates and is omitted.

LEMMA (4.5). *Let* $(N, h)$ *be a compact Riemannian manifold whose curvature is nonpositive. Assume that* $1 > \frac{m}{2p}$, *where* $m = \dim M$. *For any fixed* $\epsilon > 0$, *let* $\phi \in L_{1,2p}(M, N)$ *be a weak solution of* (4.4), *i.e., a critical point of* $E_\epsilon$ *on it. Then the supremum norm* $\| \ \|_\infty$ *of the energy density* *function* $e(\phi)$ *of the weak solution* $\phi$ *can be estimated as*

$$\|e(\phi)\|_\infty \le C\,E(\phi),$$

*where* $\|e(\phi)\|_\infty := \sup\{|e(\phi)|;\ x \in M\}$, *and the constant* $C$ *does not depend on* $\phi$ *but only on* $(M, g)$ *and* $(N, h)$.

Now to prove (ii), applying Theorem 1.11.1 in Morrey [Mo] (cf. p. 36 in [Mo]), $e(\phi)^{p/2} = (1/2)^{p/2}|d\phi|^p$ belongs to $L_{1,2}(M, \mathbf{R})$. By Lemma (4.5), it turns out that if $E(\phi)$ is sufficiently small, then $\phi \in L_{2,2p}(M, \mathbf{R}^K)$. Then (4.4) can be regarded a second order elliptic linear operator in an unknown $d\phi$ with Hörder continuous coefficients and using Schauder's theory, one can conclude $\phi \in C^\infty(M, N)$, but the details will be omitted.

(iii) Proof of (iii) of Theorem (4.1). Using Lemma (4.5), we show the following:

Proposition (4.6). *For any* $\delta > 0$ *and* $b > 0$, *set*

$$S^b_\delta := \{\phi \in L_{1,2p}(M, N);\ 0 \le \epsilon \le \delta,\ \phi \text{ is a critical point of } E_\epsilon, \text{ and } E_\epsilon(\phi) \le b\}.$$

*Then the set* $S^b_\delta$ *has a compact closure in* $L_{1,2p}(M, N)$.

Proof. From Lemma (4.5), we have

$$\|e(\phi)\|_\infty \le Cb < \infty \tag{4.7}$$

for all $\phi \in S^b_\delta$.

Now let $\{\phi_i\}_{i=1}^\infty$ be an arbitrary sequence such that $\phi_i$ is a critical point of $E_{\epsilon_i}$ satisfying $0 \le \epsilon_i \le \delta$ and $E_{\epsilon_i}(\phi_i) \le b$. Then we shall show $\{\phi_i\}_{i=1}^\infty$ has a convergent subsequence in $L_{1,2p}(M, N)$. Since $\epsilon_i \in [0, \delta]$, $i = 1, 2, \dots$, it has a convergent sequence, so we may assume $\{\epsilon_i\}_{i=1}^\infty$ is convergent. The following two cases occur:

(i) $\epsilon_i \to \epsilon > 0$ as $i \to \infty$.

(ii) $\epsilon_i \to 0$ as $i \to \infty$.

Case (i). In this case, since $d(E_{\epsilon_i})_{\phi_i} = d(E + \epsilon_i J)_{\phi_i} = 0$, $dE_{\phi_i} = -\epsilon_i dJ_{\phi_i}$. Thus, we get

$$\|d(E + \epsilon J)_{\phi_i}\| = \|(\epsilon - \epsilon_i) dJ_{\phi_i}\| \le |\epsilon - \epsilon_i|\,\|dJ_{\phi_i}\| \le C'|\epsilon - \epsilon_i|, \tag{4.8}$$

where $C'$ is a positive constant independent on $\phi_i$.

Indeed, for $\phi \in L_{1,2p}(M, N)$ and $w \in L_{1,2p}(M, \mathbf{R}^K)$,

$$\begin{aligned} |dJ_\phi(w)| &\le 2p \int_M |d\phi|^{2p-2}\,|(dw, d\phi)|\, v_g \\ &\le 2p \int_M |d\phi|^{2p-1}\,|dw|\, v_g \\ &\le 2p(Cb)^{(2p-1)/2} \int_M |dw|\, v_g \quad \text{by (4.7)} \\ &\le C'\|w\|_{1,2p} \end{aligned}$$

from Hölder's inequality (cf. (3.6) in Chapter 3). Thus, we obtain $\|dJ_\phi\| \le C'$.

Therefore, the right-hand side of (4.8) converges to zero as $i \to \infty$. So we have that

$$\inf\{d(E_\epsilon)_{\phi_i}\,;\ i = 1, 2, \dots\} = 0.$$

Since $E_\epsilon$ satisfies the condition (C) in $L_{1,2p}(M, N)$, we can choose a convergent subsequence, denoted by the same letter, of $\{\phi_i\}_{i=1}^\infty$, such that

$$\phi_i \to \phi \quad \text{as } i \to \infty,$$

in $L_{1,2p}(M, N)$ and $\phi$ is a critical point of $E_\epsilon$, which is the desired result.

Case (ii): $\epsilon_i \to 0$ as $i \to \infty$. In this case, $\epsilon_i$ can be arbitrarily small as $i \to \infty$. So we fix a small $\epsilon_i$ and denote it by $\epsilon$. We can rewrite the Euler-Lagrange equation (4.4) for $E_\epsilon = E + \epsilon J$ as follows. By (ii) in Theorem (4.1), we may assume each $\phi := \phi_i$ is smooth, and denoting $\Delta\phi = (\Delta\phi_1, \dots, \Delta\phi_K)$ for $\phi = (\phi_1, \dots, \phi_K)$,

$$\begin{aligned}\Delta\phi + p\epsilon\Bigg\{&\delta(|d\phi|^{2p-2}d\phi)\\ &+ |d\phi|^{2p-2}\sum_{i,j=1}^m A_{\phi(x)}\left(d\phi\left(\frac{\partial}{\partial x_i}\right), d\phi\left(\frac{\partial}{\partial x_j}\right)\right)g^{ij}\Bigg\}\\ + \sum_{i,j=1}^m &A_{\phi(x)}\left(d\phi\left(\frac{\partial}{\partial x_i}\right), d\phi\left(\frac{\partial}{\partial x_j}\right)\right)g^{ij} = 0.\end{aligned}$$

But this equation can be regarded as a form

$$\Delta\phi = \epsilon\, T \cdot \nabla d\phi + B, \tag{4.9}$$

where $T$, $B$ are tensors depending on $(x, \phi(x), d\phi(x))$. We may assume the sets $\|T\|_\infty$, $\|B\|_\infty$ are bounded where $\phi = \phi_i$ runs over $S_\delta^b$. Then for $\phi = \phi_i$, it follows that

$$\begin{aligned}\|\phi\|_{2,2p} &\le \|\Delta\phi\|_{2p} + \|e(\phi)\|_p + \|\phi\|_{2p}\\ &\le \epsilon\|T\|_\infty\|\phi\|_{2,2p} + \|B\|_\infty + \|e(\phi)\|_p + \|\phi\|_{2p},\end{aligned}$$

by definition of $\|\ \|_{2,2p}$ and (4.9). Since $\epsilon$ can be taken as arbitrarily small and $\|T\|_\infty$ is bounded, we can take

$$\epsilon\|T\|_\infty \le \frac{1}{2},$$

which yields

$$\left(1 - \frac{1}{2}\right)\|\phi\|_{2,2p} \le \|B\|_\infty + \|e(\phi)\|_p + \|\phi\|_{2p}.$$

Here note that

$$\begin{aligned}\|e(\phi)\|_p &\le C\,E(\phi)\,\mathrm{Vol}(M, g) \le Cb\,\mathrm{Vol}(M, g),\\ \|\phi\|_{2p} &\le \sup_M|\phi|\,\mathrm{Vol}(M, g).\end{aligned}$$

Moreover, since $\phi \in C^\infty(M, N)$ and $M$, $N$ are compact, we have that $\sup_M |\phi| \le C' < \infty$, where $C'$ does not depend on $\phi$. Thus, $\|e(\phi)\|_p + \|\phi\|_{2p}$ is uniformly bounded on $S_\delta^b$. Together with the above inequality, we can estimate

$$\|\phi\|_{2,2p} \le C < \infty.$$

That is, $\{\phi_i\}_{i=1}^\infty$ is a bounded set in $L_{2,2p}(M, N)$.

Due to Sobolev's Lemma (4.30) in Chapter 2, the inclusion

$$L_{2,2p}(M, N) \hookrightarrow L_{1,2p}(M, N)$$

is completely continuous, so $\{\phi_i\}_{i=1}^\infty$ has a convergent subsequence in $L_{1,2p}(M, N)$, We denote a convergent subsequence by the same letter as its limit $\phi$ in $L_{1,2p}(M, N)$.

In the equation

$$(dE + \epsilon_i dJ)_{\phi_i} = 0,$$

since $\phi_i \to \phi$ as $i \to \infty$, we obtain

$$\|dJ_{\phi_i} - dJ_\phi\| \le C'' \|\phi_i - \phi\|_{1,2p} \to 0,$$
$$\|dE_{\phi_i} - dE_\phi\| \le C''' \|\phi_i - \phi\|_{1,2p} \to 0,$$

and since $\epsilon_i \to 0$, we can conclude $dE_\phi = 0$.

Thus, in both cases, $\{\phi_i\}_{i=1}^\infty$ has a convergent subsequence in $L_{1,2p}(M, N)$. □

COROLLARY (4.10). *Let $(M, g)$, $(N, h)$ be compact Riemannian manifolds. Assume that the curvature of $(N, h)$ is nonpositive and $1 > \frac{m}{2p}$, $m = \dim M$. Then $E$ attains a minimum on each connected component $\mathcal{H}$ of $L_{1,2p}(M, N)$.*

PROOF. $E_\epsilon = E + \epsilon J$ satisfies the condition (C) on $L_{1,2p}(M, N)$ for all $\epsilon > 0$, so it attains a minimum on each connected component $\mathcal{H}$ of $L_{1,2p}(M, N)$ (cf. Theorem (2.17) in Chapter 3). So let $\phi_\epsilon \in \mathcal{H}$ be a minimizer of $E_\epsilon$. Then for each $\phi \in \mathcal{H}$,

$$E(\phi_\epsilon) \le E(\phi_\epsilon) + \epsilon J(\phi_\epsilon)$$
$$\le E(\phi) + \epsilon J(\phi).$$

Let $\epsilon \to \infty$. By Proposition (4.6), if we denote the accumulation point of $\{\phi_\epsilon;\ \epsilon > 0\}$ in $L_{1,2p}(M, N)$ by $\phi_0$, then $\phi_0 \in \mathcal{H}$, since $\mathcal{H}$ is a closed subset of $L_{1,2p}(M, N)$. Moreover, by the above inequality,

$$E(\phi_0) = \lim_{\epsilon \to 0} E(\phi_\epsilon) \le \lim_{\epsilon \to 0}(E(\phi) + \epsilon J(\phi)) = E(\phi),$$

so we can conclude that this $\phi_0 \in \mathcal{H}$ is a minimizer of $E$. □

The $\phi_0$ in Corollary (4.10), satisfies

$$\phi_0 \in L_{1,2p}(M, N) \subset C^0(M, N)$$

since $1 > \frac{m}{2p}$, $m = \dim M$, and it is a critical point of $E$. Thus, by Proposition (2.48), $\phi_0$ is $C^\infty$ and a harmonic mapping which minimizes the energy in its homotopy class, which implies (iii) of Theorem (4.1).

# Solutions to Exercises

**1.1.** (i) $u_{xx} = 0$. (ii) $\frac{d}{dx}(u_x{}^{p-1}) = 0$. (iii) $u_{xx} + \epsilon p \frac{d}{dx}(u_x{}^{p-1}) = 0$.

**1.2.** Putting $g(x) = \frac{1}{a}\int^x f(s)\,ds$, then $y(x) = \int^x \frac{g(s)}{\sqrt{1-g(s)^2}}\,ds$.

**1.3.** If $u(t, x) = F(t)\,G(x)$, then

$$\rho\, u_{tt} + \mu\Delta u = 0 \iff \rho\, F''(t)\, G(x) + \mu\, F(t)\,\Delta G(x) = 0$$
$$\iff \begin{cases} \rho\, F'' + \lambda F = 0 & \text{on } \mathbf{R} \\ \mu\Delta G - \lambda G = 0 & \text{on } \Omega, \end{cases}$$

where $\lambda$ is a constant. $G$ is a function on $\overline{\Omega}$ vanishing on $\partial\Omega$. Then $G$ is expanded into

$$G(x) = \sum_{i=1}^{\infty} a_n G_n(x),$$

where $\{G_n\}_{i=1}^{\infty}$ is a complete orthonormal basis of $L^2(\Omega)$ with respect to the inner product $(f_1, f_2) = \int_\Omega f_1(x) f_2(x)\,dx$ satisfying

$$\begin{cases} \mu\Delta G_n = \lambda_n G_n & \text{on } \Omega, \\ G_n(x) = 0, & x \in \partial\Omega. \end{cases}$$

Note that $\lambda_n > 0$.

Now $u(t, x)$ satisfies the boundary condition $u(t, x) = 0$, $(t, x) \in \mathbf{R} \times \partial\Omega$, and it can be expanded into

$$u(t, x) = \sum_{n=1}^{\infty} b_n(t)\, G_n(x).$$

Then we get

$$\rho\, u_{tt} + \mu\Delta u = 0 \iff \rho \sum_{n=1}^{\infty} b_n''(t)\, G_n(x) + \mu \sum_{n=1}^{\infty} b_n(t)\,\Delta G_n(x) = 0$$
$$\iff \sum_{n=1}^{\infty} (\rho b_n''(t) + \lambda_n\, b_n(t))\, G_n(x) = 0$$
$$\iff \rho\, b_n''(t) + \lambda_n\, b_n(t) = 0,$$

using $(G_m, G_n) = \delta_{mn}$. Since $u_0(x)$, $u_1(x)$ satisfy the boundary condition $u_0 = 0$, $u_1 = 0$ on $\partial\Omega$, we can expand them into

$$u_0(x) = \sum_{n=1}^{\infty} c_n G_n(x), \qquad u_1(x) = \sum_{n=1}^{\infty} d_n G_n(x).$$

The initial condition at $t = t_0$ is

$$u(t_0, x) = u_0(x) \iff \sum_{n=1}^{\infty} b_n(t_0) G_n(x) = \sum_{n=1}^{\infty} c_n G_n(x)$$
$$\iff b_n(t_0) = c_n, \qquad n = 1, 2, \dots,$$

and

$$u_t(t_0, x) = u_1(x) \iff \sum_{n=1}^{\infty} b_n'(t_0) G_n(x) = \sum_{n=1}^{\infty} d_n G_n(x)$$
$$\iff b_n'(t_0) = d_n, \quad n = 1, 2, \dots.$$

Thus, $b(t)$, $n = 1, 2, \dots$ should satisfy

$$b_n''(t) = -\frac{\lambda_n}{\rho} b_n(t), \qquad b_n(t_0) = c_n, \qquad b_n'(t_0) = d_n.$$

Such $b_n$ are given by

$$b_n(t) = \sqrt{\frac{\rho}{\lambda_n}}\, d_n \sin\left(\sqrt{\frac{\lambda_n}{\rho}}\,(t - t_0)\right) + c_n \cos\left(\sqrt{\frac{\lambda_n}{\rho}}(t - t_0)\right).$$

Substituting this into $u(x, t) = \sum_{n=1}^{\infty} b_n(t) G_n(x)$, we obtain the desired solution.

**2.1.** Use Zorn's proposition (1.12) and the formulas in (1.21).

**2.2.** We show completeness of $L(E, F)$ with respect to the norm $\| \;\|$. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Since for $x \in E$,

$$\|T_n(x) - T_m(x)\| \le \|T_n - T_m\| \, \|x\|,$$

$\{T_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $F$ and since $F$ is complete, there exists $y \in F$ to which $T_n(y)$ converges. Defining $T(x) = y$, $T$ is the desired limit. The boundedness of $T$ can be seen as follows: Since $\|T_n - T_m\| \to 0$ as $n, m \to \infty$, $\{\|T_n\|\}_{n=1}^{\infty}$ is a bounded set, say $\|T_n\| \le C$, $n = 1, 2, \dots$. As $n \to \infty$ in the inequality

$$\|T_n(x)\| \le \|T_n\| \, \|x\| \le C \, \|x\|,$$

we get $\|T(x)\| \le C\|x\|$ since $| \, \|T_n(x)\| - \|T(x)\| \, | \le \|T_n(x) - T(x)\| \to 0$.

To show $\|T_n - T\| \to 0$ as $n \to \infty$, we take any $\epsilon > 0$. Then there exists an $n_0$ such that if $n, m \ge n_0$ such that $\|T_n - T_m\| < \epsilon$. Then for $x \in E$,

$$\|T_n(x) - T_m(x)\| \le \epsilon \, \|x\|.$$

As $m \to \infty$, we get

$$\|T_n(x) - T(x)\| \leq \epsilon \|x\|.$$

Thus, if $n > n_0$, we can conclude $\|T_n - T\| \leq \epsilon$.

**2.3.** We first prepare with the following facts about the Lie derivaitve $L_X S$, $X \in \mathfrak{X}(M)$ of a $(r, s)$-tensor field $S$: Denoting by $\varphi_t(x)$ the maximal integral curve of $X \in \mathfrak{X}(M)$ through $x \in M$, $\varphi_t$ induces an isomorphism between $T_x M$ and $T_{\varphi_t(x)} M$ which induces also the one between $T_x^* M$ and $T^*_{\varphi_t(x)} M$. Then $\varphi_t$ induces an isomorphism between the tensor spaces $T_x^{r,s} M$ and $T^{r,s}_{\varphi_t(x)} M$, denoted by

$$\tilde{\varphi}_t : T_x^{r,s} M \to T^{r,s}_{\varphi_t(x)} M.$$

For a $C^\infty$-$(r, s)$-tensor field $S$, define a new $(r, s)$ tensor $L_X S \in \Gamma(T^{r,s} M)$, called the **Lie derivative** by

$$(L_X S)_x := \left.\frac{d}{dt}\right|_{t=0} \tilde{\varphi}_t^{-1} S_{\varphi_t(x)}, \qquad x \in M.$$

Then

(i) $L_X f = Xf$, $f \in C^\infty(M)$,

(ii) $L_X Y = [X, Y]$, $Y \in \mathfrak{X}(M)$,

(iii) $L_X(S \otimes T) = L_X S \otimes T + S \otimes L_X T$, for $C^\infty$ tensor fields $S$, $T$ on $M$,

(iv) $L_X(\omega \wedge \eta) = L_X \omega \wedge \eta + \omega \wedge L_X \eta$ for $C^\infty$ forms $\omega$, $\eta$ on $M$,

(v) For a $C^\infty$-$(0, s)$-tensor $\omega$ and $X_1, \dots, X_s \in \mathfrak{X}(M)$,

$$(L_X \omega)(X_1, \dots, X_s) = X(\omega(X_1, \dots, X_s)) - \sum_{i=1}^{s} \omega(X_1, \dots, [X, X_i], \dots, X_s).$$

Indeed, for (ii) take $U$, $(x_1, \dots, x_n)$ a coordinate neighborhood around $x \in M$. Denote an integral curve of $X = \sum_{i=1}^n X_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$ by $\varphi_t(x) = (\varphi_1(t, x), \dots, \varphi_n(t, x))$. It follows that

$$\frac{d}{dt} \varphi_i(t, x) = X_i(\varphi_1(t, x), \dots, \varphi_n(t, x)), \qquad \varphi_i(0, x) = x_i$$

for $1 \leq i \leq n$. Ignoring the higher order terms in $t$, the solution $\varphi_t(t, x)$ is of the form

$$\varphi_i(t, x) = x_i + X_i(x_1, \dots, x_n) t + O(t^2).$$

Since $\varphi_t(x)^{-1} = \varphi_{-t}(x)$,

$$\varphi_i(-t, x) = x_i - X_i(x_1, \dots, x_n) t + O(t^2).$$

Thus, for $Y_y = \sum_{j=1}^n Y_j(y) (\frac{\partial}{\partial x_j})_y$, $y \in U$,

$$\varphi_{-t*} Y_{\varphi_t(x)} = \sum_{j=1}^{n} Y_j(\varphi_t(x)) \varphi_{-t*} \left(\frac{\partial}{\partial x_j}\right)_{\varphi_t(x)},$$

where

$$\begin{aligned}\varphi_{-t*}\left(\frac{\partial}{\partial x_j}\right)_{\varphi_t(x)} &= \sum_{i=1}^n \left(\frac{\partial}{\partial x_j}\right)_{\varphi_t(x)} (x_i \circ \varphi_{-t})\left(\frac{\partial}{\partial x_i}\right)_x \\ &= \sum_{i=1}^n \left(\frac{\partial}{\partial x_j}\right)_{\varphi_t(x)}\left\{x_i - X_i(x_1, \dots, x_n)t + O(t^2)\right\}\left(\frac{\partial}{\partial x_i}\right)_x \\ &= \sum_{i=1}^n \left\{\delta_{ij} - \frac{\partial X_i}{\partial x_j}(x_1, \dots, x_n)t + O(t^2)\right\}\left(\frac{\partial}{\partial x_i}\right)_x,\end{aligned}$$

and

$$Y_j(\varphi_t(x)) = Y_j(x_1, \dots, x_n) + \sum_{k=1}^n \frac{\partial Y_j}{\partial x_k}(x_1, \dots, x_n)X_k(x_1, \dots, x_n)t + O(t^2).$$

So we get

$$\begin{aligned}(L_X Y)_x &= \frac{d}{dt}\bigg|_{t=0} \varphi_{-t*}Y_{\varphi_t(x)} \\ &= \sum_{j=1}^n \frac{d}{dt}\bigg|_{t=0}\left\{Y_j(\varphi_t(x))\varphi_{-t*}\left(\frac{\partial}{\partial x_j}\right)_{\varphi_t(x)}\right\} \\ &= \sum_{i=1}^n\sum_{j=1}^n \left\{\frac{\partial Y_i}{\partial x_j}(x)X_j(x) - Y_j(x)\frac{\partial X_i}{\partial x_j}(x)\right\}\left(\frac{\partial}{\partial x_i}\right)_x = [X, Y]_x\end{aligned}$$

which yields (ii). For (v), let $\omega$ be a $(0, s)$-tensor field. Then

$$\begin{aligned}(L_X\omega)_x(X_1, \dots, X_s) &= \frac{d}{dt}\bigg|_{t=0}(\varphi_t^*\omega)_x(X_1, \dots, X_s) \\ &= \frac{d}{dt}\bigg|_{t=0}\omega_{\varphi_t(x)}(\varphi_{t*}X_1, \dots, \varphi_{t*}X_s) \\ &= \frac{d}{dt}\bigg|_{t=0}\omega_{\varphi_t(x)}(X_1, \dots, X_s) \\ &\quad + \sum_{i=1}^s \omega_x\left(X_1, \dots, \frac{d}{dt}\bigg|_{t=0}\varphi_{t*}X_{i\,x}, \dots, X_s\right),\end{aligned}$$

the first term of which is equal to $X_x(\omega(X_1, \dots, X_s))$, and for $Y \in \mathfrak{X}(M)$,

$$-\frac{d}{dt}\bigg|_{t=0}\varphi_{t*}Y = \frac{d}{dt}\bigg|_{t=0}\varphi_{-t*}Y_{\varphi_t(x)} = (L_XY)_x = [X, Y]_x$$

which yields (v).

Now on any neighborhood $(U, (x_1, \dots, x_n))$ in $(M, g)$,

$$v = \sqrt{g}\,dx_1 \wedge \cdots \wedge dx_n = \omega_1 \wedge \cdots \wedge \omega_n,$$

where $\{\omega_j\}_{j=1}^n$ is a dual basis of a local orthonormal frame field $\{e_i\}_{i=1}^n$, i.e., $\omega_j(e_i) = \delta_{ij}$, defines a global $n$-form, called the **volume form** of $(M, g)$.

Then we get

$$\begin{aligned}
\operatorname{div}(X) &= \sum_{i=1}^{n} g(e_i, \nabla_{e_i} X) = \sum_{i=1}^{n} g(e_i, \nabla_X e_i - [X, e_i]) \\
&= -\sum_{i=1}^{n} g(e_i, [X, e_i]) \qquad (\text{ since } 0 = Xg(e_i, e_i) = 2g(\nabla_X e_i, e_i)) \\
&= -\sum_{i=1}^{n} \omega_i([X, e_i]) \\
&= -\sum_{i=1}^{n} \begin{vmatrix} \omega_1(e_1) & \cdots & \omega_1([X, e_i]) & \cdots & \omega_1(e_n) \\ \vdots & & \vdots & & \vdots \\ \omega_n(e_1) & \cdots & \omega_n([X, e_i]) & \cdots & \omega_n(e_n) \end{vmatrix} \quad (i\text{th column cofactor exp.}) \\
&= -\sum_{i=1}^{n} (\omega_1 \wedge \cdots \wedge \omega_n)(e_1, \dots, [X, e_i], \dots, e_n) \\
&= (L_X v)(e_1, \dots, e_n) \quad (\text{from } X(v(e_1, \dots, e_n)) = 0 \quad \text{and (v)}) \\
&= \frac{1}{\sqrt{g}} (L_X v)\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \\
&= \frac{1}{\sqrt{g}} \left\{ X\left(v\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right) \right. \\
&\qquad \left. - \sum_{i=1}^{n} v\left(\frac{\partial}{\partial x_1}, \dots, \left[X, \frac{\partial}{\partial x_i}\right], \dots, \frac{\partial}{\partial x_n}\right)\right\} \\
&= \frac{1}{\sqrt{g}} \left\{ X\sqrt{g} + \sum_{i=1}^{n} \sqrt{g} \frac{\partial X_i}{\partial x_i} \right\} \quad \left( \text{since } \left[X, \frac{\partial}{\partial x_i}\right] = -\sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i} \frac{\partial}{\partial x_j} \right) \\
&= \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\sqrt{g}\, X_i).
\end{aligned}$$

**2.4.** (i) Since $X = \operatorname{grad} f$ satisfies $g(X, Y) = Yf$, $Y \in \mathfrak{X}(M)$, $g(X, e_i) = e_i f$, $1 \le i \le n$. Thus, we get

$$X = \sum_{i=1}^{n} g(X, e_i) e_i = \sum_{i=1}^{n} (e_i f) e_i.$$

In the same way, since $g(X, \frac{\partial}{\partial x_j}) = \frac{\partial f}{\partial x_j}$, $1 \le j \le n$,

$$X = \sum_{i,j=1}^{n} g^{ij} g\left(X, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_i} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}.$$

(ii) Since $df_j = \sum_{i=1}^{n} e_i(f_j)\omega_i$, $\operatorname{grad} f_j = \sum_{i=1}^{n} e_i(f_j) e_i$, $j = 1, 2$ and $g(\omega_i, \omega_j) = g(e_i, e_j) = \delta_{ij}$, we get

$$g(df_1, df_2) = \sum_{i=1}^{n} e_i(f_1) e_i(f_2) = g(\operatorname{grad} f_1, \operatorname{grad} f_2).$$

**2.5.** By (3.24), (3.26), for $X = \operatorname{grad} f$,

$$\Delta f = -\operatorname{div}\operatorname{grad} f = -\frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x_j}\right)$$

can be written as

$$\begin{aligned}\operatorname{div}\operatorname{grad} f &= \sum_{i=1}^{n} g(e_i, \nabla_{e_i}X) = \sum_{i,j=1}^{n} g(e_i, \nabla_{e_i}((e_j f)e_j)) \\ &= \sum_{i,j=1}^{n} g(e_i, (e_i e_j f)e_j + (e_j f)\nabla_{e_i}e_j) \\ &= \sum_{i=1}^{n}\left\{e_i^2 f + \sum_{j=1}^{n}(e_j f)g(e_i, \nabla_{e_i}e_j)\right\}.\end{aligned}$$

Here since $g(e_i, \nabla_{e_i}e_j) = -g(\nabla_{e_i}e_i, e_j)$, we get

$$\sum_{j=1}^{n}(e_j f)g(e_i, \nabla_{e_i}e_j) = -\sum_{j=1}^{n}(e_j f)g(\nabla_{e_i}e_i, e_j) = -\nabla_{e_i}e_i f$$

which implies $\operatorname{div}\operatorname{grad} f = \sum_{i=1}^{n}\{e_i^2 f - \nabla_{e_i}e_i f\}$.

The equation

$$\frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x_j}\right) = \sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n}\Gamma_{ij}^{k}\frac{\partial f}{\partial x_k}\right)$$

can be obtained as follows. Since the left-hand side coincides with

$$\sum_{i,j=1}^{n}\left\{g^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}(\sqrt{g}g^{ij})\frac{\partial f}{\partial x_j}\right\},$$

it suffices to show

$$\sum_{i,j=1}^{n}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}(\sqrt{g}g^{ij})\frac{\partial f}{\partial x_j} = -\sum_{i,j,k=1}^{n} g^{ij}\Gamma_{ij}^{k}\frac{\partial f}{\partial x_k}.$$

To do this, it suffices to see

$$\sum_{i=1}^{n}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}(\sqrt{g}g^{ij}) = -\sum_{i,k=1}^{n} g^{ik}\Gamma_{ik}^{j}$$

for all $1 \le k \le n$. Here due to the differentiation rule of a determinant, we obtain

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_i} = \frac{1}{2}\frac{1}{\det(g_{ij})}\frac{\partial\det(g_{ij})}{\partial x_i} = \frac{1}{2}\sum_{k,\ell=1}^{n} g^{k\ell}\frac{\partial g_{k\ell}}{\partial x_i}.$$

Differentiate $\sum_{k=1}^{n} g_{ik}g^{kj} = \delta_{ij}$ in $x_\ell$, then we get

$$\sum_{k=1}^{n}\left\{\frac{\partial g_{ik}}{\partial x_\ell}g^{kj} + g_{ik}\frac{\partial g^{kj}}{\partial x_\ell}\right\} = 0, \qquad \frac{\partial g^{ij}}{\partial x_\ell} = -\sum_{s,t=1}^{n} g^{is}\frac{\partial g_{st}}{\partial x_\ell}g^{tj}.$$

Thus, we should show the left-hand side coincides with

$$\sum_{i=1}\left\{\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_i}g^{ij}+\frac{\partial g^{ij}}{\partial x_i}\right\}=\sum_{i,k,\ell=1}^{n}\frac{1}{2}g^{ij}g^{k\ell}\frac{\partial g_{k\ell}}{\partial x_i}-\sum_{i,s,t=1}^{n}g^{is}g^{tj}\frac{\partial g_{st}}{\partial x_i}.$$

which is equal to the right-hand side using (3.7) in a similar way to what we have done previously.

**2.6.** For $\eta\in A^1(M)$,

$$\begin{aligned}-\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)&=-\sum_{i=1}^{n}\{e_i(\eta(e_i))-\eta(\nabla_{e_i}e_i)\}\\&=-\sum_{i=1}^{n}\{e_i\,g(X,e_i)-g(X,\nabla_{e_i}e_i)\}\\&=-\sum_{i=1}^{n}g(\nabla_{e_i}X,e_i)=-\operatorname{div}(X)\end{aligned}$$

from the definition of $X$. Thus, by (i) of Proposition (3.29), we get

$$\int_M f(-\operatorname{div}(X))\,v_g=\int_M g(\operatorname{grad}f,X)\,v_g=\int_M g(df,\eta)\,v_g,$$

since $X=\sum_{i=1}^{n}\eta(e_i)e_i$ by means of $\eta(Y)=g(X,Y)$ for all $Y\in\mathfrak{X}(M)$, so that

$$g(\operatorname{grad}f,X)=\sum_{i=1}^{n}e_i(f)\eta(e_i)=\sum_{i=1}^{n}df(e_i)\eta(e_i)=g(df,\eta).$$

The above equality implies that $\delta\eta=-\operatorname{div}(X)$ by definition of $\delta\eta$.

**2.7.** The right-hand side of the equation to be proved coincides with

$$\begin{aligned}&\sum_{i=1}^{r+1}(-1)^{i+1}X_i(\omega(X_1,\ldots,\widehat{X}_i,X_{r+1}))\\&\quad-\sum_{i=2}^{r+1}(-1)^{i+1}\sum_{j=1}^{i-1}\omega(X_1,\ldots,\nabla_{X_i}X_j,\ldots,\widehat{X}_i,\ldots,X_{r+1})\\&\quad-\sum_{i=1}^{r}(-1)^{i+1}\sum_{j=i+1}^{r+1}\omega(X_1,\ldots,\widehat{X}_i,\ldots,\nabla_{X_i}X_j,\ldots,X_{r+1}).\end{aligned}$$

In the third term, changing the signs $(-1)^{i+1}$ and $\sum_{j=i+1}^{r+1}$, and the summations in $i$ and $j$ (cf. Figure 1 on next page), we get

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=i+1}^{r+1}(-1)^{i+1}\omega(X_1,\ldots,\widehat{X}_i,\ldots,\nabla_{X_i}X_j,\ldots,X_{r+1})\\&\quad=\sum_{j=2}^{r+1}\sum_{i=1}^{j-1}(-1)^{i+1}\omega(X_1,\ldots,\widehat{X}_i,\ldots,\nabla_{X_i}X_j,\ldots,X_{r+1})\\&\quad=\sum_{i=2}^{r+1}\sum_{j=1}^{i-1}(-1)^{j+1}\omega(X_1,\ldots,\widehat{X}_j,\ldots,\nabla_{X_j}X_i,\ldots,X_{r+1}).\end{aligned}$$

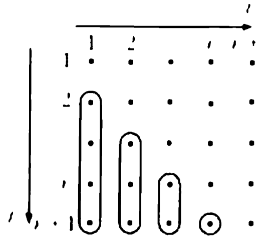


FIGURE 1

Then making use of $[X_j, X_i] = \nabla_{X_j} X_i - \nabla_{X_i} X_j$, the above is equal to

$$\begin{aligned}
&\sum_{i=1}^{r+1}(-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{r+1})) \\
&\quad - \sum_{i=2}^{r+1}(-1)^{i+1} \sum_{j=1}^{i-1} \omega(X_1, \dots, \nabla_{X_i} X_j, \dots, \widehat{X}_i, \dots, X_{r+1}) \\
&\quad - \sum_{i=2}^{r+1} \sum_{j=1}^{i-1} (-1)^{j+1} \omega(X_1, \dots, \widehat{X}_j, \dots, \nabla_{X_j} X_i, \dots, X_{r+1}) \\
&= \sum_{i=1}^{r+1}(-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{r+1})) \\
&\quad + \sum_{j<i}(-1)^{i+j} \omega([X_j, X_i], X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_i, \dots, X_{r+1}) \\
&= d\omega(X_1, \dots, X_{r+1}).
\end{aligned}$$

**2.8.** Compare two coordinates $\sum_{i=1}^n y_i v_i \mapsto (y_1, \dots, y_n)$ with respect to a basis $\{v_i\}_{i=1}^n$, and $\sum_{i=1}^n x_i e_i \mapsto (x_1, \dots, x_n)$ with respect to the standard basis $\{e_i\}_{i=1}^n$ of $\mathbf{R}^n$. Writing $v_j = \sum_{i=1}^n a_{ij} e_i$, we get

$$g_{k\ell} = \langle v_k, v_\ell \rangle = \left\langle \sum_{i=1}^n a_{ik} e_i, \sum_{j=1}^n a_{j\ell} e_j \right\rangle = \sum_{i=1}^n a_{ik} a_{i\ell},$$

$$\sum_{j=1}^n y_j v_j = \sum_{j=1}^n y_j \left( \sum_{i=1}^n a_{ij} e_i \right) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} y_j \right) e_i.$$

Thus, $x_i = \sum_{j=1}^n a_{ij} y_j$, and $dx_i = \sum_{j=1}^n a_{ij}\, dy_j$. Hence,

$$\begin{aligned}
\pi^* g_\Lambda = g_0 = \sum_{i=1}^n dx_i \otimes dx_i &= \sum_{i=1}^n \left( \sum_{k=1}^n a_{ik} dy_k \right) \otimes \left( \sum_{\ell=1}^n a_{i\ell} dy_\ell \right) \\
&= \sum_{k,\ell=1}^n \left( \sum_{i=1}^n a_{ik} a_{i\ell} \right) dy_k \otimes dy_\ell.
\end{aligned}$$

Using $\pi(\sum_{i=1}^n y_i v_i) \mapsto (y_1, \ldots, y_n)$, we get $g_\Lambda = \sum_{k,\ell=1}^n g_{k\ell}\, dy_k \otimes dy_\ell$, where $g_{k\ell} = \langle v_k, v_\ell\rangle$.

**2.9.** (i) To show $\tilde{X}$ is left invariant, i.e, $L_{a*}\tilde{X}_x = \tilde{X}_{ax}$, $a, x \in \mathrm{GL}(n, \mathbb{R})$, one may check this directly. Here we give an alternate proof. For $X \in \mathfrak{gl}(n, \mathbb{R})$, put $\tilde{X}_e = \sum_{i,j=1}^n X_{ij}\left(\frac{\partial}{\partial x_{ij}}\right)_e \in T_e\mathrm{GL}(n, \mathbb{R})$ and $\tilde{X}_a := L_{a*}\tilde{X}_e$. It suffices to show that $\tilde{X}_a$ is given by (4.12). To do this, we only have to see $\tilde{X}_a x_{ij} = \sum_{k=1}^n a_{ik} X_{kj}$. Since $(x_{ij} \circ L_a)(b) = x_{ij}(ab) = \sum_{k=1}^n x_{ik}(a) x_{kj}(b) = \sum_{k=1}^n a_{ik} b_{kj}$, we have

$$\tilde{X}_a x_{ij} = L_{a*}\tilde{X}_e x_{ij} = \tilde{X}_e(x_{ij} \circ L_a) = \sum_{s,t,k=1}^n a_{ik} X_{st} \delta_{ks} \delta_{tj} = \sum_{k=1}^n a_{ik} X_{kj}.$$

(ii) To show $[\tilde{X}, \tilde{Y}] = [X, Y]^\sim$, by definition (2.22), we only have to show

$$[\tilde{X}, \tilde{Y}]_a = \sum_{i,j,k=1}^n a_{ik} [X, Y]^\sim_{kj} \frac{\partial}{\partial x_{ij}}$$

for $X = (X_{ij})$, $Y = (Y_{ij})$, where $[X, Y]_{kj}$ is $(k, j)$-component of $[X, Y] = XY - YX$.

(iii) A curve $\sigma(t) = \exp(t\tilde{X})$ in $\mathrm{GL}(n, \mathbb{R})$ is by definition a solution of

$$\sigma'(t) = \tilde{X}_{\sigma(t)}, \ \sigma(0) = I \Longleftrightarrow \frac{d}{dt} x_{ij}(\sigma(t)) = \sum_{k=1}^n x_{ik}(\sigma(t))\, X_{kj}, \ \sigma(o) = I.$$

Considering $Y(t) = e^{tX}$ (which is convergent), we have $\frac{d}{dt}Y(t) = Y(t)X$, $Y(0) = I$. Therefore, by a uniqueness theorem for the initial value problem, we get $\sigma(t) = Y(t)$.

**2.10.** $\mathrm{GL}(n, \mathbb{C}) := \{Z \in M(n, \mathbb{C});\ \det Z \neq 0\}$ is an open subset in $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ and so becomes a Lie group of real dimension $2n^2$, by checking that the multiplication of matrices and the inverse operation are $C^\infty$-mappings with respect to the differential structure induced from $\mathbb{R}^{2n^2}$.

A set of $n$ column vectors of any element $x = (x_{ij})$ in $\mathrm{O}(n)$ is a orthonormal basis $(\mathbb{R}^n, \langle\ ,\ \rangle)$, and then $|x_{ij}| \leq 1$, $1 \leq i, j \leq n$. Thus, both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are bounded closed subsets in $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$, i.e., compact. By a similar argument, all $z = (z_{ij}) \in \mathrm{U}(n)$ satisfy $|z_{ij}| \leq 1$, $1 \leq i, j \leq n$. So both $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact subsets in $\mathrm{GL}(n) \subset \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Since any closed subgroup $K$ of a Lie group $G$ is also a Lie subgroup of $G$, we get the desired result.

To find the corresponding Lie subalgebras, it suffices to show that the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ is $\mathfrak{gl}(n\mathbb{C}) = M(n, \mathbb{C})$ and one may make use of (4.7) and exercise 2.9.

**4.1.** If $\phi : (M, g) \to (N, h)$ is an onto isometry, then $\tau(\phi) = 0$ since $\phi$ satisfies that ${}^N\nabla_{\phi_* X} \phi_* Y = \phi_* \nabla_X Y$, $X, Y \in \mathfrak{X}(M)$.

**4.2.** If $\{e_1, e_2\}$ is a local orthonormal frame field, $\{\omega_1, \omega_2\}$ is its dual, $e_g(\phi)$ is the energy density function of $\phi$ with respect to $g$, and $v_g$ is the volume element, then

$$e_g(\phi)\, v_g = \sum_{i=1}^{2} \phi^* h(e_i, e_i)\, \omega_1 \wedge \omega_2.$$

On the other hand, since $\left\{\frac{1}{\sqrt{\lambda}} e_1, \frac{1}{\sqrt{\lambda}} e_2\right\}$ is a local orthonormal frame field of $(M, \lambda g)$ and its dual is $\{\sqrt{\lambda}\omega_1, \sqrt{\lambda}\omega_2\}$, we get

$$\begin{aligned} e_{\lambda g}(\phi)\, v_{\lambda g} &= \sum_{i=1}^{2} \phi^* h\left(\frac{1}{\sqrt{\lambda}} e_1, \frac{1}{\sqrt{\lambda}} e_2\right) \sqrt{\lambda}\omega_1 \wedge \sqrt{\lambda}\omega_2 \\ &= \sum_{i=1}^{2} \phi^* h(e_1, e_2)\omega_1 \wedge \omega_2 = e_g(\phi)\, v_g \end{aligned}$$

which implies the desired result.

**4.3.** Take $\eta \in C^\infty(M)$ with $\operatorname{supp}(\eta) \subset U$. Define a $C^\infty$ mapping $F_t : U \to U$ by $F_t(x, y) := (x + t\eta(x, y), y)$, $(x, y) \in U$, extend it to the whole $M$ by $F_t(p) = p \notin U$, and define a variation $\phi_t$ of $\phi$ by $\phi_t := \phi \circ F_t$.

Then changing the coordinate on $U$ by $(\zeta, \tau) = F_t(x, y)$, we get

$$\begin{aligned} \frac{\partial \phi_t}{\partial x} &= \frac{\partial \phi}{\partial \zeta}(F_t(x, y))\left(1 + t\frac{\partial \eta}{\partial x}\right), \\ \frac{\partial \phi_t}{\partial y} &= \frac{\partial \phi}{\partial \zeta}(F_t(x, y))\, t\frac{\partial \eta}{\partial y} + \frac{\partial \phi}{\partial \tau}(F_t(x, y)). \end{aligned}$$

By exercise 4.2, we get

$$E(\phi_t) = \int_U \left\{\left|\frac{\partial \phi_t}{\partial x}\right|^2 + \left|\frac{\partial \phi_t}{\partial y}\right|^2\right\} dx\, dy + \int_{M\setminus U} e_g(\phi)\, v_g.$$

The second term does not depend on $t$, and the first term coincides with

$$\int_U \left\{\left|\frac{\partial \phi}{\partial \zeta}\right|^2 \left(1 + t\frac{\partial \eta}{\partial x}\right)^2 + \left|\frac{\partial \phi}{\partial \zeta} t \frac{\partial \eta}{\partial y} + \frac{\partial \phi}{\partial \tau}\right|^2\right\} \frac{d\zeta d\tau}{1 + t\frac{\partial \eta}{\partial x}}.$$

Therefore, we get

$$\begin{aligned} 0 = \left.\frac{dE(\phi_t)}{dt}\right|_{t=0} &= \int_U \left\{\left(\left|\frac{\partial \phi}{\partial \zeta}\right|^2 2\frac{\partial \eta}{\partial x} + 2\left\langle \frac{\partial \phi}{\partial \zeta}, \frac{\partial \phi}{\partial \tau}\right\rangle \frac{\partial \eta}{\partial y}\right)\right. \qquad (1) \\ &\qquad \left. + \left(\left|\frac{\partial \phi}{\partial \zeta}\right|^2 + \left|\frac{\partial \phi}{\partial \tau}\right|^2\right)\left(-\frac{\partial \eta}{\partial x}\right)\right\} d\zeta\, d\tau \\ &= \int_U \left\{\left(\left|\frac{\partial \phi}{\partial \zeta}\right|^2 - \left|\frac{\partial \phi}{\partial \tau}\right|^2\right)\frac{\partial \eta}{\partial x} + 2\left\langle \frac{\partial \phi}{\partial \zeta}, \frac{\partial \phi}{\partial \tau}\right\rangle \frac{\partial \eta}{\partial y}\right\} d\zeta\, d\tau \\ &= \int_U \left\{\left(\left|\frac{\partial \phi}{\partial x}\right|^2 - \left|\frac{\partial \phi}{\partial y}\right|^2\right)\frac{\partial \eta}{\partial x} + 2\left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle \frac{\partial \eta}{\partial y}\right\} dx\, dy, \end{aligned}$$

since $(x, y) = (\zeta, \tau)$ at $t = 0$.

By a similar argument, considering $G_t(x, y) = (x, y + t\eta(x, y))$, defining a variation $\phi_t := \phi \circ G_t$ of $\phi$, and changing the coordinate as $(\zeta, \tau) = G_t(x, y)$, we get

$$\frac{\partial \phi_t}{\partial x} = \frac{\partial \phi}{\partial \zeta}(G_t(x, y)) + \frac{\partial \phi}{\partial \tau}(G_t(x, y))\, t \frac{\partial \eta}{\partial x},$$
$$\frac{\partial \phi_t}{\partial y} = \frac{\partial \phi}{\partial \tau}(G_t(x, y))\left(1 + t\frac{\partial \eta}{\partial y}\right),$$

so we get

$$E(\phi_t) = \int_U \left\{\left|\frac{\partial \phi_t}{\partial x}\right|^2 + \left|\frac{\partial \phi_t}{\partial y}\right|^2\right\} dx\, dy + \int_{M\setminus U} e_g(\phi)\, v_g,$$

where the first term coincides with

$$\int_U \left\{\left|\frac{\partial \phi}{\partial \zeta} + t\frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial \tau}\right|^2 + \left|\frac{\partial \phi}{\partial \tau}\left(1 + t\frac{\partial \eta}{\partial y}\right)\right|^2\right\} \frac{d\zeta\, d\tau}{1 + t\frac{\partial \eta}{\partial y}}.$$

Thus, by an argument similar to (1), we get

$$\begin{aligned} 0 = \left.\frac{dE(\phi_t)}{dt}\right|_{t=0} &= \int_U \left\{2\left\langle\frac{\partial \phi}{\partial \zeta}, \frac{\partial \phi}{\partial \tau}\right\rangle \frac{\partial \eta}{\partial x} + 2\frac{\partial \eta}{\partial y}\left|\frac{\partial \phi}{\partial \tau}\right|^2\right) \\ &\qquad - \left(\left|\frac{\partial \phi}{\partial \zeta}\right|^2 + \left|\frac{\partial \phi}{\partial \tau}\right|^2\right)\frac{\partial \eta}{\partial y}\right\} d\zeta\, d\tau \\ &= \int_U \left\{-\frac{\partial \eta}{\partial y}\left(\left|\frac{\partial \phi}{\partial x}\right|^2 - \left|\frac{\partial \phi}{\partial y}\right|^2\right) + 2\left\langle\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle\frac{\partial \eta}{\partial x}\right\} dx\, dy. \end{aligned} \tag{2}$$

Carrying out the partial integrals, since $\eta$ is arbitrary, we obtain

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \phi}{\partial x}\right|^2 - \left|\frac{\partial \phi}{\partial y}\right|^2\right) + 2\frac{\partial}{\partial y}\left(\left\langle\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle\right) = 0 \quad (\text{by } (1)),$$
$$\frac{\partial}{\partial y}\left(\left|\frac{\partial \phi}{\partial x}\right|^2 - \left|\frac{\partial \phi}{\partial y}\right|^2\right) - 2\frac{\partial}{\partial x}\left(\left\langle\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle\right) = 0 \quad (\text{by } (2)),$$

which imply that the real and imaginary parts of $\psi$ satisfy Cauchy-Riemann equations, and then $\psi$ is holomorphic in $z = x + \sqrt{-1}y$.

REMARK. $\psi\, dz \otimes dz$ gives a globally defined holomorphic quadratic form on a Riemann surface $M$. If $M = S^2$, all holomorphic quadratic forms vanish, so that $\psi = 0$, and we obtain

$$\left|\frac{\partial \phi}{\partial x}\right|^2 - \left|\frac{\partial \phi}{\partial y}\right|^2 = 0, \qquad \left\langle\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle = 0.$$

**4.4.** (i) Let $\iota_1 : M_1 \ni x \mapsto (x, y) \in M_1 \times M_2$, $\iota_2 : M_2 \ni y \mapsto (x, y) \in M_1 \times M_2$, $\pi_1 : M_1 \times M_2 \ni (x, y) \mapsto x \in M_1$, and $\pi_2 : M_1 \times M_2 \ni (x, y) \mapsto y \in M_2$. Then we get

$$\tau(\phi) = \pi_{1*}\tau(\phi \circ \iota_1) + \pi_{2*}\tau(\phi \circ \iota_2).$$

By assumption $\tau(\phi \circ \iota_1) = 0$ and $\tau(\phi \circ \iota_2) = 0$, and we obtain $\tau(\phi) = 0$.

(ii) $\phi$ is an isometry in each variable, so it is harmonic in each variable. We may apply (i).

(iii) It suffices to show $S^{p-1} \ni x \mapsto F(x, y) \in S^{n-1}$ and $S^{q-1} \ni y \mapsto F(x, y) \in S^{n-1}$ are both harmonic. The former is linear in $x$ and then each $\phi^i(x)$ is also linear in $x$, i.e., an eigenfunction of the Laplacian on $(S^{p-1}, g_{S^{p-1}})$ with the eigenvalue $p-1$. We denote here $\iota \circ \phi = (\phi^1, \dots, \phi^n)$, where $\iota : S^{n-1} \subset \mathbb{R}^n$ is the inclusion. By a theorem of Takahashi (2.4) in Chapter 4 or (1.9) in Chapter 6, it is harmonic. That the latter is harmonic is shown in the same way.

**4.5.** Show that Hopf's mapping $\phi : (S^3, g_{S^3}) \to (S^2, g_{S^2})$ is a Riemannian submersion for which $x \in S^2$, $\phi^{-1}(x)$ is a geodesic, i.e., a minimal submanifold of $(S^3, g_{S^3})$. By Proposition (3.10), $\phi$ is harmonic.

**5.1.** (i) Since $\Delta_H = \overline{\Delta} + \rho$, $J_{\mathrm{id}} = \overline{\Delta} - \rho = 2\overline{\Delta} - \Delta_H$. Putting $\omega(Y) = g(X, Y)$, $\forall Y \in \mathfrak{X}(M)$,

$$\begin{aligned}\int_M g(J_{\mathrm{id}}X, X)\, v_g &= 2\int_M g(\overline{\Delta}X, X)\, v_g - \int_M g(\Delta_H X, X)\, v_g \qquad (1)\\ &= 2\int_M |\nabla X|^2 v_g - \int_M \mathrm{div}(X)^2 v_g - \int_M |d\omega|^2 v_g,\end{aligned}$$

and $d\omega(Z, W) = (\nabla_Z\omega)(W) - (\nabla_W\omega)(Z)$. The Lie derivation of $L_X g$ satisfies

$$\begin{aligned}(L_X g)(Z, W) &= X \cdot g(Z, W) - g([X, Z], W) - g(Z, [X, W])\\ &= g(\nabla_X Z, W) + g(Z, \nabla_X W) - g(\nabla_X Z - \nabla_Z X, W)\\ &\quad - g(Z, \nabla_X W - \nabla_W X)\\ &= g(\nabla_Z X, W) + g(Z, \nabla_W X)\\ &= Z \cdot g(X, W) - g(X, \nabla_Z W) + W \cdot g(Z, X) - g(\nabla_W Z, X)\\ &= Z \cdot \omega(W) - \omega(\nabla_Z W) + W \cdot \omega(Z) - \omega(\nabla_W Z)\\ &= (\nabla_Z\omega)(W) + (\nabla_W\omega)(Z).\end{aligned}$$

So putting $\nabla\omega(Z, W) := (\nabla_Z\omega)(W)$, we get $\nabla\omega = \frac{1}{2}\{d\omega + L_X g\}$. Then

$$|\nabla\omega|^2 = \frac{1}{2}|d\omega|^2 + \frac{1}{4}|L_X g|^2, \qquad (2)$$

since

$$|\nabla\omega|^2 = \sum_{i,j=1}^m (\nabla_{e_i}\omega)(e_j)^2,$$

$$|d\omega|^2 = \sum_{i<j} d\omega(e_i, e_j)^2 = \frac{1}{2}\sum_{i,j=1}^m d\omega(e_i, e_j)^2,$$

$$|L_X g|^2 = \sum_{i,j=1}^m L_X g(e_i, e_j)^2.$$

On the other hand, we get

$$|\nabla\omega|^2 = |\nabla X|^2, \tag{3}$$

from the calculation

$$\begin{aligned}|\nabla\omega|^2 &= \sum_{i,j=1}^{m} (\nabla_{e_i}\omega)(e_j)^2 = \sum_{i,j=1}^{m} \{e_i(\omega(e_j)) - \omega(\nabla_{e_i}e_j)\}^2 \\ &= \sum_{i,j=1}^{m} \{e_i \cdot g(X, e_j) - g(X, \nabla_{e_i}e_j)\}^2 = \sum_{i,j=1}^{m} g(\nabla_{e_i}X, e_j)^2 \\ &= \sum_{i=1}^{m} g(\nabla_{e_i}X, \nabla_{e_i}X) = |\nabla X|^2.\end{aligned}$$

Therefore, substituting (2), (3) into (1), we obtain the desired equality.

(ii) Due to the Kodaira-de Rham-Hodge decomposition, we get the orthogonal direct decomposition

$$A^1(M) = \{\omega \in A^1(M);\ \delta\omega = 0\} \oplus \{df;\ f \in C^\infty(M)\},$$
$$\mathfrak{X}(M) = \{X \in \mathfrak{X}(M);\ \operatorname{div} X = 0\} \oplus \{\operatorname{grad} f;\ f \in C^\infty(M)\}.$$

If $(M, g)$ is Einstein, i.e., $\rho = cI$, then

$$J_{\mathrm{id}} = \Delta_H - 2cI,$$

which preserves the two subspaces of $\mathfrak{X}(M)$, $\{X \in \mathfrak{X}(M);\ \operatorname{div} X = 0\}$, $\{\operatorname{grad} f;\ f \in C^\infty(M)\}$ invariantly. In fact, if $\operatorname{div} X = 0$, $\operatorname{div}(\Delta_H X) = 0$ since putting $g(X, Y) = \omega(Y)$, $\forall Y \in \mathfrak{X}(M)$, we get $g(\Delta_H X, Y) = (\Delta_1\omega)(Y)$, and $\operatorname{div}(\Delta_H X) = -\delta\Delta_1\omega = -\delta d\delta\omega = -\Delta(\delta\omega) = -\Delta(\operatorname{div} X) = 0$. Furthermore, $\Delta_H \operatorname{grad} f = \operatorname{grad}(\Delta f)$ since $g(\Delta_H \operatorname{grad} f, Y) = \Delta_1 df(Y) = d\delta df(Y) = d(\Delta f)(Y) = g(\operatorname{grad}(\Delta f), Y)$ for all $Y \in \mathfrak{X}(M)$.

Case (i). For $X \in \mathfrak{X}(M)$ with $\operatorname{div} X = 0$,

$$\int_M g(J_{\mathrm{id}}X, X)\,v_g = \int_M \left\{\frac{1}{2}|L_X g|^2 - (\operatorname{div} X)^2\right\} v_g = \frac{1}{2}\int_M |L_X g|^2 v_g \geq 0$$

which implies that the eigenvalues of $J_{\mathrm{id}}$ are nonnegative on the subspace $\{X \in \mathfrak{X}(M);\ \operatorname{div} X = 0\}$.

Case (ii). On the subspace $\{\operatorname{grad} f;\ f \in C^\infty(M)\}$, we get

$$J_{\mathrm{id}} \operatorname{grad} f = (\Delta_H - 2cI)\operatorname{grad} f = \operatorname{grad}(\Delta f) - 2c \operatorname{grad} f.$$

Here expanding $f \in C^\infty(M)$ into the eigenfunctions of $\Delta$, we get

$$f = \sum_{i=1}^{\infty} f_i, \quad \Delta f_0 = 0, \quad \Delta f_i = \lambda_i f_i, \qquad i \geq 1.$$

By the above, we obtain

$$J_{\mathrm{id}} \operatorname{grad} f_i = (\lambda_i - 2c)\operatorname{grad} f_i, \qquad i \geq 1$$

which yields that eigenvalues of $J_{\mathrm{id}}$ on the subspace $\{\operatorname{grad} f;\ f \in \mathfrak{X}(M)\}$ are $\{\lambda_i - 2c;\ i \geq 1\}$.

Putting both cases together, we obtain that index(id) $= 0 \iff \lambda_1(g) = \lambda_1 \geq 2c$.

**5.2.** Since $J_{\mathrm{id}} = \overline{\Delta} - \rho$, we get

$$0 = \int_M g(J_{\mathrm{id}} X, X)\, v_g = \int_M g(\overline{\Delta} X, X)\, v_g - \int_M g(\rho(X), X)\, v_g$$
$$= \int_M |\nabla X|^2 v_g - \int_M g(\rho(X), X)\, v_g.$$

By the assumption $\rho \leq 0$, we get $0 \leq \int_M |\nabla X|^2 v_g = \int_M g(\rho(X), X)\, v_g \leq 0$. Thus, we obtain $\nabla X = 0$.

**5.3.** For $\omega \in A^1(M)$, define $X \in \mathfrak{X}(M)$ by $\omega(Y) = g(X, Y)$ for all $Y \in \mathfrak{X}(M)$. We get $\Delta_1 \omega = \overline{\Delta}\omega + \rho(X)$, where $\overline{\Delta}\omega(Y) = g(\overline{\Delta} X, Y)$ for all $Y \in \mathfrak{X}(M)$. Thus, for $f \in C^\infty(M)$ with $\Delta f = \lambda_1 f$,

$$\begin{aligned}\|\nabla df\|^2 + (\operatorname{grad} f, \rho(\operatorname{grad} f)) &= (df, \overline{\Delta}(df)) + (\operatorname{grad} f, \rho(\operatorname{grad} f))\\ &= (df, \Delta_1 df) = (df, d\Delta f) = (\delta df, \Delta f)\\ &= (\Delta f, \Delta f) = \lambda_1 (f, \Delta f) = \lambda_1 \|df\|^2.\end{aligned}$$

By the assumption on $\rho$, we get

$$(\operatorname{grad} f, \rho(\operatorname{grad} f)) \geq \alpha(\operatorname{grad} f, \operatorname{grad} f) = \alpha \|df\|^2.$$

Thus, we get

$$0 \geq \left(-1 + \frac{\alpha}{\lambda_1}\right)\|\Delta f\|^2 + \|\nabla df\|^2, \tag{1}$$

since $\|\Delta f\| = \lambda_1 \|df\|^2$. On the other hand,

$$\|\nabla df\|^2 \geq \frac{1}{n}\|\Delta f\|^2 \tag{2}$$

since putting $h := \nabla df$, for $X$, $Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}h(X, Y) &= \nabla_X(df)(Y) = X(df(Y)) - df(\nabla_X Y)\\ &= X(Yf) - (\nabla_X Y)f = Y(Xf) + [X, Y]f - (\nabla_X Y)f\\ &= Y(Xf) - (\nabla_Y X)f = \nabla_Y(df)(X) = h(Y, X)\end{aligned}$$

which implies that $h$ is a symmetric (0,2) tensor field. Moreover, define

$$\operatorname{tr} h := \sum_{i=1}^n h(e_i, e_i) \sum_{i=1}^n \nabla_{e_i}(df)(e_i) = \sum_{i=1}^n \{e_i{}^2 f - \nabla_{e_i} e_i f\} = -\Delta f,$$
$$|h|^2 := \sum_{i,j=1}^n h(e_i, e_j)^2.$$

By Schwarz' inequality, denoting $h_{ij} := h(e_i, e_j)$,

$$\left(\sum_{i,j=1}^n h_{ij}{}^2\right) n = \left(\sum_{i,j=1}^n h_{ij}{}^2\right)\left(\sum_{i,j=1}^n \delta_{ij}\right) \geq \left(\sum_{i,j=1}^n h_{ij}\delta_{ij}\right)^2 = \left(\sum_{i=1}^n h_{ii}\right)^2.$$

Thus, we obtain

$$|\nabla df|^2 \geq \frac{1}{n}(\mathrm{tr}\ \nabla df)^2 = \frac{1}{n}|\Delta f|^2.$$

Integrating this over $M$, we obtain (2).

From (1) and (2), we obtain

$$0 \geq \left(\frac{1}{n} - 1 + \frac{\alpha}{\lambda_1}\right)\|\Delta f\|^2$$

which implies $0 \geq \frac{1}{n} - 1 + \frac{\alpha}{\lambda_1}$ since $\|\Delta f\| > 0$. We obtain the desired result.

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